ALGEBRA PRELIM—AUGUST 2008

Solve eight of the twelve problems below. If you provide solutions (full or partial) to more that eight problems, clearly indicate which eight should be graded.

Group theory.

(1) Show that the map $[\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3] \mapsto [\mathbf{a}_2 \times \mathbf{a}_3\ \ \mathbf{a}_3 \times \mathbf{a}_1\ \ \mathbf{a}_1 \times \mathbf{a}_2]$ is an endomorphism on $\mathrm{GL}(3, \mathbb{R})$. (Here, $\mathrm{GL}(3, \mathbb{R})$ denotes the multiplicative group of invertible 3×3 matrices with real entries, and $[\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3]$ denotes the matrix with columns $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$.)

(2) Let G be a finite group, and let N be a normal subgroup of G. Let p be a prime, and let P denote a p-Sylow subgroup of N. Assume P is normal in N. Show that P is normal in G.

(3) Show that a group of order $1309 = 7 \cdot 11 \cdot 17$ is cyclic.

(4) Let G be a group, and assume that $g \mapsto g^2$ is an endomorphism on G. Show that G is Abelian.

Ring theory and modules. All rings are assumed to be commutative, and to have an identity element 1.

(5) Let R be a ring. Assume that for any two principal ideals Ra and Rb we have either $Ra \subseteq Rb$ or $Rb \subseteq Ra$. Show that for any two ideals I and J in R, we have either $I \subseteq J$ or $J \subseteq I$.

(6) Consider the subring $R = \mathbb{Z}[2x, 2x^2, 2x^3, \dots]$ of the polynomial ring $\mathbb{Z}[x]$. Let $I = (2x)$ and $J = (2x^2)$ be ideals in R. Show that the ideal $I \cap J$ in R is not finitely generated. Conclude that R is not factorial.

(7) Let R be an integral domain, and let M be an R-module. Show that

$$M' = \{m \in M \mid rm = 0 \text{ for some non-zero } r \text{ in } R\}$$

is a submodule of M.

(8) Let R be a ring, and let $I \subseteq R$ be a finitely generated ideal. Show that if $I^2 = I$ then $I = Re$ where $e \in R$ is an idempotent element (i.e., $e^2 = e$).

Fields and Galois theory.

(9) Find all finite fields F with the property that if F' and F'' are subfields of F then either $F' \subseteq F''$ or $F'' \subseteq F'$.

(10) Prove that if $f(x) \in \mathbb{Q}[x]$ is an irreducible quartic whose discriminant has a rational square root, then the Galois group of $f(x)$ over $\mathbb{Q}$ has order 4 or 12.

(11) Let F be a finite field and $F(\alpha, \beta)/F$ a finite extension. Prove that if $F(\alpha) \cap F(\beta) = F$ then the composite of $F(\alpha)$ and $F(\beta)$ is $F(\alpha + \beta)$.

(12) Let M be the splitting field for $x^8 + 2$ over $\mathbb{Q}$. Find $[M : \mathbb{Q}]$.