ALGEBRA PRELIM—MAY 2008

Solve eight of the twelve problems below. If you provide solutions (full or partial) to more that eight problems, clearly indicate which eight should be graded.

Group theory.

(1) Let G be a finite Abelian group. A *group character* of G is a homomorphism χ from G to the multiplicative group of non-zero complex numbers $\mathbb{C}^*$. Let $\widehat{G}$ denote the set of group characters of G. Show that $\widehat{G}$ is a group with the operation of pointwise multiplication, i.e., $\chi_1\chi_2$ is the map sending g to $\chi_1(g)\chi_2(g)$. Then prove: If G is cyclic then $G \simeq \widehat{G}$.

(2) Let n be an odd natural number ≥ 3, and let D_n denote the dihedral group of order $2n$. Show that $D_{2n} \simeq D_n \times C_2$.

(3) Show that a group of order $2057 = 11^2 \cdot 17$ is Abelian.

(4) Let G be a group in which $g^2 = 1$ for all elements g. Show that G is Abelian.

Ring theory and modules. All rings are assumed to be commutative, and to have an identity element 1.

(5) Let R be a ring, and let $\operatorname{Nil}(R)$ be its nilradical. Show that the following three statements are equivalent:

(i) R contains exactly one prime ideal.
(ii) Any element in R is either invertible or nilpotent.
(iii) The ring $R/\operatorname{Nil}(R)$ is a field.

(6) Let R be a ring, and let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals in R with $\mathfrak{a}+\mathfrak{b} = R$ and $\mathfrak{a}\cap\mathfrak{b} = \{0\}$. Show that there exists an element $e \in R$ with $\mathfrak{a} = Re$, $\mathfrak{b} = R(1-e)$, and $e^2 = e$.

(7) Show that the ideal in $\mathbb{Z}[x]$ generated by $x^2 + 1$ and $x^3 + x^2 + 1$ is principal.

(8) The $\mathbb{Q}$-algebra $\mathbb{Q}[x, y]/(x^2, y^2)$ is a graded $\mathbb{Q}$-vector space of finite dimension. Determine the dimension.

Fields and Galois theory.

(9) Let $f(x) = x^5 - 4x + 2 \in \mathbb{Q}[x]$. Show that the roots of $f(x)$ cannot be expressed in terms of radicals over $\mathbb{Q}$.

(10) Let $p(x) = x^4 + ax^2 + 1 \in \mathbb{Q}[x]$ and assume that $p(x)$ is irreducible over $\mathbb{Q}$. Let θ be a root of $p(x)$. Show that $\mathbb{Q}(\theta)$ is the splitting field of $p(x)$, and that both $(\theta + 1/\theta)^2$ and $(\theta - 1/\theta)^2$ are in $\mathbb{Q}$. Compute the Galois group of $\mathbb{Q}(\theta)/\mathbb{Q}$.

(11) Let K be a field, let $f(x) \in K[x]$ be a polynomial of degree p, where p is a prime, and let M be the splitting field. Prove that if $\operatorname{Gal}(M/K) \simeq C_p$, then $f(x)$ is irreducible.

(12) Let p be a prime. Show that $\mathbb{F}_{p^n}$ is a subfield of $\mathbb{F}_{p^m}$ if and only if n divides m.