ALGEBRA PRELIM—AUGUST 2009

Solve eight (8) of the twelve (12) problems below. If you provide solutions (full or partial) to more that eight problems, clearly indicate which eight should be graded.

Group theory

(1) Give examples of groups of the following type:
  (a) an infinite non-abelian group,
  (b) an infinite torsion group,
  (c) a subgroup that is not normal,
  (d) a subgroup that is normal,
  (e) a group homomorphism that is neither injective nor surjective.

(2) Consider the group $\mathsf{GL}(2, \mathbb{F}_2)$ of invertible 2×2 matrices with entries from the field of two elements.
  (a) Show that this group has 6 elements.
  (b) Show that it is isomorphic to the symmetric group on 3 letters.

(3) Let G be the set of all rational functions p/q in $\mathbb{R}(t)$ for which $\deg(p) < \deg(q)$ (with the convention that $\deg(0) = -\infty$), and define an operation $*$ by

$$f * g = f + g + fg.$$

Show that $(G, *)$ is a torsion-free Abelian group.

(4) Let G be a group, and assume that $G/Z(G)$ is cyclic. Show that G is Abelian.

Ring theory and modules

(5) Let $\mathbf{i} = \sqrt{-1} \in \mathbb{C}$, and let x be an indeterminate.
  (a) Show that the sets $\mathbb{Z} \times \mathbb{Z}$, $\mathbb{Z}[\mathbf{i}]$, and $\mathbb{Z}[x]/(x^2)$ considered as additive groups are isomorphic.
  (b) Show that they are pairwise non-isomorphic as rings.

(6) Let R be a commutative ring and M be a nonzero finitely generated R-module.
  (a) Prove that the set of proper submodules of M has maximal elements.
  (b) Let N be a maximal submodule of M. Show that M/N is isomorphic as an R-module to $R/\mathfrak{m}$, where $\mathfrak{m} \subset R$ is a maximal ideal of R.

(7) Let R be any ring and a be an element of R.
  (a) Prove that if a has a left inverse, then a is not a left 0-divisor.
  (b) Prove that the converse holds if $a \in aRa$.

(8) Prove that a finite ring is a field if and only if it is an integral domain.

Fields and Galois theory

(9) Let $R \subset \mathbb{Z}[x]$ be the subring of polynomials without linear or quadratic terms. Show that the field of fractions of R is $\mathbb{Q}(x)$.

(10) Let $\mathbb{F}$ be a field. Let α and β be algebraic over $\mathbb{F}$. Denote their minimal polynomials by f and g respectively.
   (a) Show that g is irreducible over $\mathbb{F}(\alpha)$ if and only if f is irreducible over $\mathbb{F}(\beta)$.
   (b) Let $\mathbb{K}$ be the splitting field of $x^4 - 6$ over $\mathbb{Q}$. Determine $|\mathbb{K} : \mathbb{Q}|$.

(11) Let K and L be two different C_4-extensions of $\mathbb{Q}$ inside $\mathbb{C}$. Assume that they have the same quadratic subextension. Find $\mathrm{Gal}(KL/\mathbb{Q})$.

(12) Let K be a field. A *derivation* on K is a map $D\colon K \to K$ satisfying

$$D(a+b) = D(a) + D(b) \quad \text{and} \quad D(ab) = D(a)b + aD(b)$$

for all $a, b \in K$. Show that

$$\mathrm{Tr}(A^{-1}D(A)) = \frac{D(\det A)}{\det A}$$

for all invertible $n \times n$ matrices A over K. (Here, $D(A)$ means that D has been applied to each entry in A.)