ALGEBRA PRELIM—AUGUST 2010

Solve eight (8) of the twelve (12) problems below. If you provide solutions (full or partial) to more than eight problems, clearly indicate which eight should be graded.

Group theory

(1) Let G be a group of order $p^n q$, where $p > q$ are primes. Show that G contains a unique subgroup of index q.

(2) Let $\sigma \in A_n$, where $n \geq 2$. Show that if σ commutes with an odd permutation in S_n, then the conjugacy classes of σ in A_n and S_n are the same.

(3) An automorphism φ of a group G is called *inner* if there exists a $g \in G$ such that

$$\forall h \in G \colon \varphi(h) = ghg^{-1}.$$

Let $\mathrm{GL}(2, \mathbb{R})$ denote the multiplicative group of non-singular 2×2 matrices over $\mathbb{R}$. Show that φ, given by

$$\varphi(A) = (A^{-1})^T,$$

is an automorphism of $\mathrm{GL}(2, \mathbb{R})$, but that it is *not* inner.

(4) A group is called *perfect* if it has trivial center and every automorphism of the group is inner (cf. (3) above). Let G be a group, and let N be a normal subgroup. Assume that N is perfect. Show that there exists a subgroup H of G, such that G is the direct product of N and H.

Ring theory and modules

(5) Set

$$R = \mathbb{Q}[x, y]/(x^3 - 2, y^2 - 5).$$

Find a $\mathbb{Q}$-basis for R, and find the matrix representation of the endomorphism given by $\psi(\alpha) = (x + y)\alpha$ with respect to that basis.

(6) Set

$$R = \{a_0 + a_1 x + a_2 \frac{x^2}{2} + \cdots + a_n \frac{x^n}{n!} \mid n \in \mathbb{N},\ a_0, \ldots, a_n \in \mathbb{Z}\}.$$

Show that R is a subring of the polynomial ring $\mathbb{Q}[x]$. Then show that R is not Noetherian.

(7) Let R be a ring. Let M and N be R-modules. Show that $M \oplus N$ is injective if and only if M and N are injective.

(8) Let $(R, +, \cdot)$ be a ring. Show that the set R endowed with the addition '+' and multiplication '×', defined by

$$a \times b = b \cdot a,$$

is a ring.

Fields and Galois theory

(9) Let $\alpha \in \mathbb{C}$. Show that $\mathbb{Z}[\alpha]$ is a $\mathbb{Z}$-module of finite rank if and only if there is a monic polynomial $f(x) \in \mathbb{Z}[x]$ with $f(\alpha) = 0$.

(10) An algebraic field extension L/K is called *purely inseparable* if the only elements in L that are separable over K are the elements of K itself. Let L/K be a field extension in prime characteristic p. Show that L/K is purely inseparable if and only if

$$\forall a \in L \; \exists n \in \mathbb{N} \colon a^{p^n} \in K.$$

(11) Determine the splitting field and its degree over $\mathbb{Q}$ for $x^6 - 9$.

(12) Find a primitive generator for $\mathbb{Q}(\sqrt[3]{5}, \zeta_3)$, where ζ_3 is a primitive third root of unity.