ALGEBRA PRELIM—MAY 2010

Solve eight (8) of the twelve (12) problems below. If you provide solutions (full or partial) to more than eight problems, clearly indicate which eight should be graded.

Group theory

(1) Let G be a group and H and K two subgroups. Show that $HK = KH$ if and only if HK is a subgroup of G.

(2) Let G be a group of prime power order p^m. Show that G contains a normal subgroup of order p.

(3) Let G be the free group on the generators $\{a, b\}$, and let N be the normal subgroup generated by aba and $a^{16}b^5$. Show that G/N is Abelian.

(4) How many groups are there of order $1729 = 7 \cdot 13 \cdot 19$.

Ring theory and modules

(5) Let R be a ring. Let M, N be R-modules. Show that $M \oplus N$ is projective if and only if M and N are projective.

(6) Let k be a field and let V be an infinite dimensional k-vector space. Prove that the endomorphism ring $R = \mathrm{End}(V)$ is isomorphic as an R-module to R^4.

(7) Let $R \neq 0$ be a commutative ring. Prove that if every ideal in R is free as an R-module, then R is a PID.

(8) Let R be a commutative ring with 1. Prove the following statement: The sum of any two principal ideals of R is principal if and only if every finitely generated ideal of R is principal.

Fields and Galois theory

(9) Let $f(x) = x^4 - 5x^3 + 5x^2 + 5x - 5 \in \mathbb{Q}[x]$. Show: If θ is a root of $f(x)$, then so is $\theta^2 - 2\theta$. Use this to determine $\mathrm{Gal}(f/\mathbb{Q})$.

(10) Let $\mathbb{F}$ be a field of characteristic not 2. Let $a, b \in \mathbb{F}$ be two elements and consider the splitting field $\mathbb{K}$ of the polynomial $(x^2 - a)(x^2 - b)$. Assume that neither a, b, nor ab are perfect squares in $\mathbb{F}$. Show that the Galois group $\mathrm{Gal}(\mathbb{K} : \mathbb{F}) \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ is isomorphic to the Klein-Four-Group.

(11) Let K be a field, and let $f \in K[x_1, \ldots, x_n]$ be a polynomial in n variables. Let S be an infinite subset of K, and assume that

$$f(s_1, \ldots, s_n) = 0$$

for all $s_1, \ldots, s_n \in S$. Show that $f = 0$.

(12) Find the splitting field for $x^4 - 10x^2 + 1$ over $\mathbb{Q}$.