ALGEBRA PRELIM—AUGUST 2011

Solve eight (8) of the twelve (12) problems below. If you provide solutions (full or partial) to more that eight problems, clearly indicate which eight should be graded.

Group theory

(1) Let G be a group of order $7021 = 7 \cdot 17 \cdot 59$. Show that G is cyclic.

(2) Let G be a group and let $f \colon G \to X$ be a function.
  (a) Show that
$$H = \{h \in G \mid \forall g \in G \colon f(hgh^{-1}) = f(g)\}$$
  is a subgroup of G.
  (b) Verify the equality
$$H = \{h \in G \mid \forall g \in G \colon f(gh) = f(hg)\}.$$
  (c) Show: If G is finite of prime power order > 1, then H is a non-trivial subgroup of G.

(3) Let G be a non-abelian group. Show that $\mathrm{Aut}(G)$ is not cyclic.

(4) Let H and K be subgroups of a finite group G. Show that the set
$$\{hk \mid h \in H, k \in K\}$$
has exactly $|H||K|/|H \cap K|$ elements.

Ring theory and modules

(5) Let x be an indeterminate.
  (a) Let F be a field of characteristic 0. Show: If $f(x), g(x) \in F[x]$ satisfy
$$f(x+1) - f(x) = g(x+1) - g(x),$$
  then one has $f(x) = g(x) + C$ for an element $C \in F$.
  (b) Set
$$R = \{f \in \mathbb{R}[x] \mid \forall a \in \mathbb{Z} \colon f(a) \in \mathbb{Z}\}.$$
  Show that R is a subring of $\mathbb{R}[x]$.
  (c) For $n \in \mathbb{N}$, set
$$\binom{x}{n} = \frac{x(x-1)\cdots(x-n+1)}{n!}.$$
  With R as in (2), verify the equality
$$R = \left\{ a_0 + a_1\binom{x}{1} + \cdots + a_n\binom{x}{n} \;\middle|\; n \in \mathbb{N},\ a_0, \ldots, a_n \in \mathbb{Z} \right\}.$$
  [*Hint:* for $f \in R$, use induction on $\deg f$ together with (2) and the identity $\binom{x+1}{n} - \binom{x}{n} = \binom{x}{n-1}$.]

(6) Let p be a prime number and let $(\mathbb{Z}/(p))^{\mathbb{N}}$ be the ring of sequences $(a_n)_{n\in\mathbb{N}}$ of elements from $\mathbb{Z}/(p)$, with coordinate-wise addition and multiplication. Assume $\mathfrak{p}$ is a prime ideal in $(\mathbb{Z}/p)^{\mathbb{N}}$. Show that one has $(\mathbb{Z}/(p))^{\mathbb{N}}/\mathfrak{p} \cong \mathbb{Z}/(p)$.

(7) Let R be a commutative ring and let M be an R-module that contains no submodules other than 0 and M. Show that there exists an ideal I in R such that $M \cong R/I$ as R-modules. [*Hint:* an element of M can be used to define a map $R \to M$.]

(8) Let R be a commutative ring. For every ideal $I \subseteq R$ set
$$(0 : I) = \{r \in R \mid rI = 0\}.$$
Prove the following assertions.
(a) $(0 : I)$ is an ideal.
(b) For ideals $I \subset J$ one has $(0 : J) \subseteq (0 : I)$.
(c) There is an equality $(0 : (0 : I)) = I$.

Fields and Galois theory

(9) Let K be a field of characteristic $\neq 2$ and let $p, q \in K$. Show that
$$f(x) = x^4 - 2(p^2 + q^2)x^2 + p^2(p^2 + q^2)$$
is irreducible in $K[x]$ if and only if $p^2 + q^2$ is not a square in K, and that the Galois group for $f(x)$ over K is then cyclic of order 4.

(10) Let $f(x) = x^4 - 10x + 20 \in \mathbb{Q}[x]$. Show that the Galois group for $f(x)$ over $\mathbb{Q}$ is cyclic of order 4.

(11) Let M/K be a Galois extension with Galois group G, and let Λ be a ring containing M as a subfield. Then Λ is a vector space over M with respect to left-multiplication. Suppose that, for each $\sigma \in G$, we have a non-zero element u_σ in Λ such that
$$\forall a \in M \colon u_\sigma a = \sigma(a) u_\sigma.$$
Show that the elements u_σ, $\sigma \in G$, are linearly independent over M.

(12) Find the minimal polynomial for $\sqrt{-3} + \sqrt{2}$ over $\mathbb{Q}$.