ALGEBRA PRELIM—MAY 2011

Solve eight (8) of the twelve (12) problems below. If you provide solutions (full or partial) to more that eight problems, clearly indicate which eight should be graded.

Group theory

(1) Let G be a group of order $1225 = 5^2 \cdot 7^2$. Show that G is abelian.

(2) Let G be a group and let

$$N = \{\sigma \in \mathrm{Aut}(G) \mid \forall g \in G\colon \sigma(g)g^{-1} \in Z(G)\}.$$

Show that N is a normal subgroup of $\mathrm{Aut}(G)$.

(3) Let G be a group of order 30. Show that G contains a cyclic subgroup of index 2.

(4) Let C_n denote the cyclic group of order n, and let D_n denote the dihedral group of order $2n$. Show that a group of order 70 is isomorphic to exactly one of the groups C_{70}, D_{35}, $C_5 \times D_7$, or $D_5 \times C_7$.

Ring theory and modules

(5) Let F be a field, and let n be a natural number. Let $x_1, \dots, x_n$ be indeterminates and let $a_1, \dots, a_n, b_1, \dots, b_n \in F$. Assume that $a_i \neq b_i$ for some index i. Show that

$$\mathfrak{a} = \{f \in F[x_1, \dots, x_n] \mid f(a_1, \dots, a_n) = 0 = f(b_1, \dots, b_n)\}$$

is an ideal in $F[x_1, \dots, x_n]$ and show that there is an isomorphism

$$F[x_1, \dots, x_n]/\mathfrak{a} \cong F \oplus F.$$

(6) Let R be a commutative ring with 1 and let F and F' be flat R-modules. Show that the module $F \otimes_R F'$ is flat.

(7) Let R be a commutative ring with 1, and let $M_n(R)$ denote the ring of $n \times n$ matrices over R. Show that the two-sided ideals in $M_n(R)$ are exactly $M_n(I)$, where I is an ideal in R.

(8) Let R be a commutative ring with 1 and let $\mathfrak{m} \subset R$ be a maximal ideal. Show that if $I \subset R$ is a proper ideal containing $\mathfrak{m}^n$ for some $n \geq 1$, then $\mathfrak{m}$ is the only prime ideal that contains I.

Fields and Galois theory

(9) Compute the Galois group over $\mathbb{Q}$ of the polynomial $x^6 - 3$.

(10) Let $F(x)$ be the quotient field of the polynomial ring in one indeterminate x over a field F. Prove that every element $r(x) = p(x)/q(x)$ that is not in F is transcendental over F. What can you say about the extension $F(x)/F(r)$?

(11) Let $p > q$ be primes. Let G be a non-abelian group of order pq. If E/F is a Galois extension with Galois group G, discuss the subfields of E.

(12) Let a be an integer and set

$$f(x) = x^3 + ax^2 + (a-3)x - 1.$$

Show: If $\theta \in \mathbb{C}$ is a root of $f(x)$, then so is $-1/(1+\theta)$. Then show that $f(x)$ is irreducible in $\mathbb{Q}[x]$, and determine the Galois group for $f(x)$ over $\mathbb{Q}$.