Algebra Preliminary Examination
August 2012

Work any eight problems. Clearly indicate which eight are to be graded.

1. Prove:
   a) If A and B are normal subgroups of G such that G/A and G/B are both abelian then $G/(A \cap B)$ is abelian.

   b) If K is a normal subgroup of G then K', the commutator subgroup of K, is a normal subgroup of G as well.

2. Let G be a group of order $1183 = 7 \cdot 13^2$. Show that G is abelian if and only if G has an element of order $91 = 7 \cdot 13$.

3. Is it possible for a group of order 12 to have three 2-Sylow subgroups and four 3-Sylow subgroups? Justify your answer by giving either an example of such a group or a proof that no such group exists.

4. Show that the polynomial $x^4 + 7x^2 - 3$ is irreducible over $\mathbb{Q}$ and determine its Galois group.

5. (a) Give an example of a commutative ring R with ideals $I \neq J$ such that R/I and R/J are isomorphic as rings.
   (b) Let R be a commutative ring and let I and J be ideals in R. Show that if R/I and R/J are isomorphic as R-modules, then one has $I = J$.

6. Let R be a commutative ring, let I be an ideal in R, and let M be an R-module. Set
$$(0 :_M I) = \{m \in M \mid Im = 0\}.$$
   (a) Show that $(0 :_M I)$ is a submodule of M.
   (b) Show that there is an isomorphism of R-modules $\operatorname{Hom}_R(R/I, M) \cong (0 :_M I)$.

7. Let R be a commutative ring (with identity) containing $\mathbb{Q}$ and let $\mathfrak{N}(R)$ be the set of nilpotent elements of R; that is
$$\mathfrak{N}(R) = \{x \in R \mid x^n = 0 \text{ for some } n \in \mathbb{Z}^+\}.$$
   Show that
$$\exp\colon x \mapsto \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

is a well-defined injective group homomorphism from $\mathfrak{N}(R)$ (as a group under addition) to the group R^* of units in R (with multiplication as the group operation).

8. Consider the commutative ring $R = \mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\}$.
   (a) Show that the subset $I = \{a + b\sqrt{-5} \mid a \equiv b \mod 2\}$ is an ideal in R.
   (b) Show that the assignment $(r, s) \mapsto (2r + (1+\sqrt{-5})s, 2s + (1-\sqrt{-5})r)$ defines an isomorphism $R \oplus R \cong I \oplus I$.
   (c) Show that I is not principal and conclude that I is a non-free projective R-module.
9. Let $K/\mathbb{Q}$ be a Galois extension with cyclic Galois group of order 3, and let σ be a generator for $\mathrm{Gal}(K/\mathbb{Q})$. Show: If α is an element in K, such that $\alpha\sigma(\alpha)$ is not a square in K, then

$$K(\sqrt{\alpha\sigma(\alpha)}, \sqrt{\sigma(\alpha)\sigma^2(\alpha)})/\mathbb{Q}$$

   is a Galois extension with Galois group A_4.

10. Let $\mathbb{K} \subseteq \mathbb{L} \subseteq \mathbb{M}$ be a tower of fields. Let $\alpha \in \mathbb{M}$ be algebraic over $\mathbb{K}$. Denote its minimal polynomial by minpol_α. Prove that $\mathbb{L} \otimes_{\mathbb{K}} \mathbb{K}[\alpha]$ and $\mathbb{L}[x]/(\mathrm{minpol}_\alpha)$ are isomorphic as $\mathbb{L}$-algebras.

11. Let $f(x) = x^3 + x^2 - 4x + 1$. Show that $f(x)$ is irreducible, and that if α is a root, then so is $2 - 2\alpha - \alpha^2$. Then use this fact to compute the Galois group of the splitting field.

12. Let K be an infinite field. Prove that if F/K is a finite separable field extension, then there is an element u of F such that $F = K(u)$.