Algebra Preliminary Examination

May 2012

Work any eight problems. Clearly indicate which eight are to be graded.

1. Let G be a group in which every element has order at most 2.
   (a) Show that G is Abelian.
   (b) Show that the order of G, if finite, is a power of 2.

2. Let p be a prime. Show that there is no simple group of order $8p$.

3. Let G be a finite group. Show that the number of conjugacy classes in G is greater than or equal to $[G : G']$, with equality if and only if G is abelian. As usual, G' denotes the commutator subgroup of G.

4. Let G be a group of order $8 \cdot 7^m$ for some $m \in \mathbb{N}$.
   (a) Show that the number of 7-Sylow subgroups of G is 1 or 8.
   (b) Let $\phi : G \longrightarrow \Sigma_8$ be a group homomorphism into the symmetric group on 8 letters. Show that the image of ϕ has strictly less than 60 elements.

5. List all prime ideals in the ring $\mathbb{Z}[x]/(30, x^2 + 1)$. List each ideal exactly once and indicate which ones are maximal.

6. Suppose that S is a unique factorization domain. Let R be a subring of S with the following property: If $s \in S$ and $r \in R$ such that s divides r, then s is an element of R. Show that R is a unique factorization domain.

7. Let R be a commutative ring and let $\mathfrak{N}(R)$ be the set of nilpotent elements of R, that is

$$\mathfrak{N}(R) = \{x \in R \mid x^n = 0 \text{ for some } n \in \mathbb{Z}^+\}$$

   (a) Show that $\mathfrak{N}(R)$ is an ideal of R.
   (b) Show that 0 is the only nilpotent element of $R/\mathfrak{N}(R)$.

8. Let R be a commutative ring. Assume that for any two principal ideals (a) and (b) in R one has $(a) \subseteq (b)$ or $(b) \subseteq (a)$. Show that for any two ideals I and J in R one has $I \subseteq J$ or $J \subseteq I$.

9. Let K be a field, and let $f(x)$ and $g(x)$ be non-constant polynomials in $K[x]$ with $\gcd(f, g) = 1$. Show: For every $h(x) \in K[x]$ such that

$\deg(h) < \deg(f) + \deg(g)$, there exists unique $p(x), q(x) \in K[x]$ with $\deg(p) < \deg(f)$, $\deg(q) < \deg(g)$ and

$$\frac{h(x)}{f(x)g(x)} = \frac{p(x)}{f(x)} + \frac{q(x)}{g(x)}.$$

Here $\deg(f)$ denotes the degree of the polynomial f.

10. What is the Galois group of $p(x) = x^3 - x + 4$, considered over the ground fields
    (a) $\mathbb{Z}/3\mathbb{Z}$,
    (b) $\mathbb{R}$,
    (c) $\mathbb{Q}$?
    Justify your answers.
11. Let $\mathbb{F}$ be a field, and let α, β be algebraic over $\mathbb{F}$. Denote their minimal polynomials by minpol_α, minpol_β, respectively. Show that minpol_α is irreducible over $\mathbb{F}(\beta)$ if and only if minpol_β is irreducible over $\mathbb{F}(\alpha)$.
12. Prove that the polynomial $p(x) = x^4 - 3x^2 - 3$ is irreducible over $\mathbb{Q}$ and compute its Galois group.