Algebra, August 2013

Work two problems from each section, i.e., eight problems altogether. Clearly indicate which eight are to be graded. Otherwise, we will grade 1,2,4,5,7,8,10, and 11.

GROUPS

PROBLEM 1:

Let p and q be odd primes with $p < q$. Show that every group of order $2pq$ has a subgroup of index 2.

PROBLEM 2:

Classify the groups of order 28. (There are four isomorphism types.)

PROBLEM 3:

Prove that D_{8n} is not isomorphic to $D_{4n} \times \mathbb{Z}/2\mathbb{Z}$. ($D_k$ denotes the dihedral group of order k.)

RINGS

PROBLEM 4:

Consider the subring $\mathbb{Z}[2X, 2X^2, 2X^3, \ldots] \subseteq \mathbb{Z}[X]$ of the ring of all polynomials with integer coefficients. Let $I = (2X)$ and $J = (2X^2)$ be two ideals. Show that $I \cap J \subseteq \mathbb{Z}[2X, 2X^2, 2X^3, \ldots]$ is not finitely generated. Conclude that $\mathbb{Z}[2X, 2X^2, 2X^3, \ldots]$ is not factorial.

PROBLEM 5:

$R = \mathbb{Z} \times \mathbb{Z}$ is a ring with addition and multiplication defined by

$$(a, b) + (c, d) = (a + c, b + d), \qquad (a, b)(c, d) = (ac + ad + bc, bd).$$

Show that there are no non-zero nilpotents in this ring.

PROBLEM 6:

Let $\mathbb{Q}[\![x]\!]$ denote the ring of formal power series over the rational numbers. Show that the only ring endomorphism $\varphi\colon \mathbb{Q}[\![x]\!] \to \mathbb{Q}[\![x]\!]$ with $\varphi(x) = x$ is the identity.

MODULES

PROBLEM 7:

Let R be a ring and let

$$\begin{array}{ccccc} M' & \xrightarrow{\alpha'} & M & \xrightarrow{\alpha} & M'' \\ \downarrow{\scriptstyle \varphi'} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi''} \\ N' & \xrightarrow{\beta'} & N & \xrightarrow{\beta} & N'' \end{array}$$

be a commutative diagram of R-modules with exact rows.

1. Show that if α is surjective, then the sequence $\mathrm{Coker}\varphi' \to \mathrm{Coker}\varphi \to \mathrm{Coker}\varphi''$ is exact.
2. Show that if β' is injective, then the sequence $\mathrm{Ker}\varphi' \to \mathrm{Ker}\varphi \to \mathrm{Ker}\varphi''$ is exact.

PROBLEM 8:

Let $0 \longrightarrow K \longrightarrow P \xrightarrow{\phi} M \longrightarrow 0$ and $0 \longrightarrow L \longrightarrow Q \xrightarrow{\psi} M \longrightarrow 0$ be exact sequences of R-modules, where P and Q are projective. Prove that

$$P \oplus L \cong Q \oplus K$$

as R-modules.

PROBLEM 9:

Let R be a commutative ring. Let M be an R-module and let $M_1,\ M_2 \subseteq M$ be submodules. Show that the canonical inclusion

$$M/(M_1 \cap M_2) \longrightarrow M/M_1 \oplus M/M_2$$

is an isomorphism of R-modules if and only if $M = M_1 + M_2$.

FIELDS

PROBLEM 10:

Show that $\mathbb{Q}(\sqrt{5+2\sqrt{6}})/\mathbb{Q}$ is a Galois extension, and determine the Galois group.

PROBLEM 11:

Let $\mathbb{K}/\mathbb{F}$ be an algebraic field extension. Let R be an intermediate ring, i.e., $\mathbb{F} \subseteq R \subseteq \mathbb{K}$. Show that R is a field.

PROBLEM 12:

Let $\mathbb{F}$ be a field of characteristic not equal to two.

1. Consider
$$\mathbb{K} = \mathbb{F}(\sqrt{D_1}, \sqrt{D_2}),$$
where $D_1, D_2 \in \mathbb{F}$ have the property that none of D_1, D_2, or $D_1 D_2$ is a square in $\mathbb{F}$. Prove that $\mathbb{K}/\mathbb{F}$ is a Galois extension with Galois group isomorphic to the Klein 4-group.

2. Conversely, assume $\mathbb{K}/\mathbb{F}$ is a Galois extension with Galois group equal to the Klein 4-group. Prove that $\mathbb{K} = \mathbb{F}(\sqrt{D_1}, \sqrt{D_2})$, where $D_1, D_2 \in \mathbb{F}$ have the property that none of D_1, D_2, or $D_1 D_2$ is a square in $\mathbb{F}$.