Algebra, May 2013

Work two problems from each section, i.e., eight problems altogether. Clearly indicate which eight are to be graded. Otherwise, we will grade 1,2,4,5,7,8,10, and 11.

GROUPS

PROBLEM 1:

Let G be a group of order $21p$, where p is a prime > 7. Show that G has a normal subgroup of index 3.

PROBLEM 2:

Let p and q be primes with $p \mid q+1$ and p odd. Show that any two subgroups of $\mathrm{GL}(2,q)$ of order p are conjugate.

PROBLEM 3:

Let n be a natural number, and let $0 \leq i \leq n$. Show that the number of subgroups of the n-fold cartesian product C_2^n of order 2^i is equal to the number of subgroups of C_2^n of order 2^{n-i}. (Here, C_2 denotes the cyclic group of order 2.)

RINGS

PROBLEM 4:

Consider the commutative ring $R = \mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\}$.

1. Show that the subset $I = \{a + b\sqrt{-5} \mid a = b \mod 2\}$ is an ideal.
2. Show that the assignment $(r, s) \mapsto (2r + (1+\sqrt{-5})s, 2s + (1-\sqrt{-5})r)$ defines an isomorphism $R \oplus R \cong I \oplus I$.
3. Show that I is not principal and conclude that I is a non-free projective R-module.

PROBLEM 5:

Consider the ring of all real-valued continuous functions defined on the interval $[0, 1]$. Let R be the subring consisting of all functions f such that

$f(0) = f(1)$. Let M be the R-module consisting of functions f such that $f(0) = -f(1)$. Prove that

1. $M \oplus M \cong R \oplus R$,
2. M is not a free R-module.

PROBLEM 6:

A ring R is called von Neumann regular if for every element x in R there is an element r such that $x = xrx$.

1. Show that every division ring is von Neumann regular.
2. Show that the product of any family $\{R_u\}_{u\in U}$ of von Neumann regular rings is von Neumann regular.

MODULES

PROBLEM 7:

Let R be a ring and consider commutative diagrams of R-modules with exact rows

$$\begin{array}{ccccccc} & & M' & \xrightarrow{\alpha'} & M & \xrightarrow{\alpha} & M'' \\ & & & & \downarrow{\varphi} & & \downarrow{\varphi''} \\ 0 & \longrightarrow & N' & \xrightarrow{\beta'} & N & \xrightarrow{\beta} & N'' \end{array} \quad \text{and} \quad \begin{array}{ccccccc} M' & \xrightarrow{\alpha'} & M & \xrightarrow{\alpha} & M'' & \longrightarrow & 0 \\ \downarrow{\psi'} & & \downarrow{\psi} & & & & \\ N' & \xrightarrow{\beta'} & N & \xrightarrow{\beta} & N'' & & \end{array}$$

Show that there exist unique homomorphisms $\varphi' \colon M' \to N'$ and $\psi'' \colon M'' \to N''$, such that the diagrams remain commutative.

PROBLEM 8:

Let R be a commutative ring.

1. Let E be a free R-module with basis $\{e_u\}_{u\in U}$, where U is finite. Show that the functionals e_u^* given by $e_v \mapsto \delta_{uv}$ form a basis for the dual module $\mathrm{Hom}_R(E, R)$.
2. Show that for every finitely generated projective R-module P, the dual module $\mathrm{Hom}_R(P, R)$ is projective as well.

3. Show that for every projective R-module the natural homomorphism $P \to \mathrm{Hom}_R(\mathrm{Hom}_R(P, R), R)$ is injective, and that it is an isomorphism if P is finitely generated.

PROBLEM 9:

Let R be a commutative and Noetherian ring. Show that for every finitely generated R-module M, the dual module $\mathrm{Hom}_R(M, R)$ is finitely generated.

FIELDS

PROBLEM 10:

Show that $\mathbb{Q}(\sqrt{5+\sqrt{5}})/\mathbb{Q}$ is a Galois extension, and determine the Galois group.

PROBLEM 11:

Let L/K be a finite field extension. Show that there exists a K-algebra homomorphism $L \to M_n(K)$ if and only if $[L : K] \mid n$.

PROBLEM 12:

Let p be a prime. Prove or disprove: There exists $p \times p$ matrices X and Y over $\mathbb{F}_p$ with $XY - YX = I$.