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ALGEBRA PRELIM—AUGUST 2014

Work two problems from each of the four sections, i.e., eight in total. Clearly indicate which two problems from each section are to be graded. Otherwise, we will grade problems 1, 2, 4, 5, 7, 8, 10 & 11. In grading, the problems will be weighted equally.

All rings are assumed to have a 1-element.

Groups

Problem 1: Let G be a group of order $105 = 3 \cdot 5 \cdot 7$. Show that G has a normal subgroup of index 3.

Problem 2: Show: For all natural numbers n, $16^{5^{n-1}} - 1$ is divisible by 5^n, but not by 5^{n+1}. Then show that $(\mathbb{Z}/5^n)^*$ is cyclic, with $\bar{2}$ as a generator. (Here, $\mathbb{Z}/5^n$ denotes the ring of integers modulo 5^n.)

Problem 3: Let p be an odd prime, and assume that $q = 2p + 1$ is also a prime. Determine—up to isomorphism—all non-Abelian groups of order $2pq$.

Rings

Problem 4: Let R be a commutative ring, and let $D\colon R \to R$ be a *derivation* on R, i.e., a map satisfying $D(a + b) = D(a) + D(b)$ and $D(ab) = D(a)b + aD(b)$. Let $\varphi\colon R \to F$ be a ring homomorphism (preserving the 1-element), where F is a field. Show that

$$\bigcap_{n=0}^{\infty} \ker(\varphi \circ D^n)$$

is a prime ideal in R.

Problem 5: Let R be a commutative ring in prime characteristic p. Show that a unit u in R has p-power multiplicative order if and only if $u - 1 \in \operatorname{nilrad}(R)$.

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Problem 6: Let F be a field, and let n be a natural number. Show that $\Phi\colon \mathrm{M}_n(F) \otimes_F \mathrm{M}_n(F) \to \mathrm{End}_F(F^{n\times n})$, given by

$$\Phi(A \otimes B)\colon X \mapsto AXB^T,$$

is a well-defined isomorphism of F-algebras.

Modules

Problem 7: (a) Show: If A is a torsion-free $\mathbb{Z}$-module, then there exists an injective homomorphism from A into a torsion-free injective $\mathbb{Z}$-module.

(b) Show: If A is a torsion $\mathbb{Z}$-module, then there exists an injective homomorphism from A into an injective torsion $\mathbb{Z}$-module.

Problem 8: Let R be a commutative ring, and let I and J be ideals in R. Assume that $I \otimes_R J \simeq R$ (as R-modules). Show that I is projective.

Problem 9: Let R be a ring. A left R-module N is called *irreducible* if $N \neq (0)$ and the only submodules of N are (0) and N. Show: If—for a given left R-module M—there exists irreducible submodules $N_1, \dots, N_s$ with $M = N_1 + \cdots + N_s$, then M is the direct sum of a subset of the N_i's.

Fields

Problem 10: Let $f(x) = x^4 - x^3 - 4x^2 + 4x + 1 \in \mathbb{Q}[x]$. You are given the following information: If θ is a root of $f(x)$, then so is $\theta^2 - 2$. Determine the Galois group for $f(x)$ over $\mathbb{Q}$.

Problem 11: Let K/F be a finite separable field extension in prime characteristic $p \neq 0$, and let L/F be a field extension with the following property: For all $\alpha \in L$ there exists an $n \in \mathbb{N}$, such that $\alpha^{p^n} \in F$. Show that $K \otimes_F L$ is a field.

Problem 12: Let F be a field of characteristic $\neq 2$, and let $a, b \in F$. The polynomial

$$f(x) = x^4 - 2(a+b)x^2 + (a-b)^2 \in F[x]$$

has $\sqrt{a} + \sqrt{b}$ as a root. Show: $f(x)$ is irreducible if and only if none of the elements a, b and ab is a square in F.