ALGEBRA PRELIM—MAY 2014

Work two problems from each of the four sections, i.e., eight in total. Clearly indicate which problems are to be graded. Otherwise, we will grade the first two from each section, i.e., problems 1, 2, 4, 5, 7, 8, 10 & 11. In grading, the problems will be weighted equally.

All rings are assumed to have a 1-element.

Groups

Problem 1: Let G be a group of order $1089 = 3^2 \cdot 11^2$. Show: If G contains an element of order 9, then it also contains an element of order 33.

Problem 2: Let p be a prime, and let $R = \mathbb{F}_p[x]/(x^{p+1})$. Show that $R^* \simeq \mathbb{Z}/(p-1) \times (\mathbb{Z}/p)^{p-2} \times \mathbb{Z}/p^2$. (Here, $\mathbb{Z}/n$ denotes the integers modulo n.)

Problem 3: Let p be an odd prime $\equiv 2 \pmod 3$. Determine—up to isomorphism—all non-Abelian groups of order $3p^2$.

Rings

Problem 4: Let R be a ring containing $\mathbb{Q}$, and generated over $\mathbb{Q}$ by two elements x and y with $yx - xy = 1$. (Such a ring exists.) Show that R is simple, i.e., has no two-sided ideals other than (0) and R.

Problem 5: Let R be a commutative ring. Describe explicitly all maps $f\colon R \times R \to R$ such that

$$\Lambda = \left\{ \begin{pmatrix} x & f(x,y) \\ 0 & y \end{pmatrix} \mid x, y \in R \right\}$$

is a subring of $\mathrm{M}_2(R)$.

Problem 6: Let R be an integral domain. A non-zero ideal I in R is called *invertible* if there exists a non-zero ideal J, such that IJ is a principal ideal in R. Show: An invertible ideal is finitely generated.

Modules

Problem 7: Let R be a ring. If M is a left R-module, then $M^2 = M \times M$ is a left $\mathrm{M}_2(R)$-module, with the scalar multiplication given by

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = \begin{pmatrix} a_{11}m_1 + a_{12}m_2 \\ a_{21}m_1 + a_{22}m_2 \end{pmatrix}.$$

(This is known, and need not be proved.)

(a) Show: If $\varphi\colon M \to N$ is a left R-module homomorphism, then $\varphi^2\colon M^2 \to N^2$, given by

$$\varphi^2 \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = \begin{pmatrix} \varphi(m_1) \\ \varphi(m_2) \end{pmatrix},$$

is a left $\mathrm{M}_2(R)$-module homomorphism.

(b) Show: If $\Phi\colon M^2 \to N^2$ is a left $\mathrm{M}_2(R)$-module homomorphism (for left R-modules M and N), then there exists a left R-module homomorphism $\varphi\colon M \to N$ with $\Phi = \varphi^2$.

Problem 8: Let $R = \mathbb{Z}[\sqrt{-5}]$, and let $P = (2, 1 + \sqrt{-5}) \subseteq R$. Show: $P \otimes_R P \simeq R$ by $a \otimes b \mapsto \frac{1}{2}ab$.

Problem 9: Let E be an injective $\mathbb{Z}$-module. Show that E is the direct sum of a torsion module and a $\mathbb{Q}$-vector space.

Fields

Problem 10: Let $f(x) = x^6 - 7x^4 + 14x^2 - 7 \in \mathbb{Q}[x]$. You are given the following information:

(i) If θ is a root of $f(x)$, then so is $\theta^3 - 3\theta$.

(ii) $(x^3 - 3x)^3 - 3(x^3 - 3x)$ gives a remainder of $-x^5 + 5x^3 - 5x$ when divided by $f(x)$.

Determine the Galois group for $f(x)$ over $\mathbb{Q}$.

Problem 11: Let K/F be a finite separable field extension, and let L/F be an arbitrary field extension. Show that $K \otimes_F L$ is isomorphic to a direct product of finitely many extension fields of L, all of which are finite separable over L.

Problem 12: Determine the minimal polynomial for $\sqrt{2} + \sqrt{3}$ over each of the following fields: (a) $\mathbb{Q}$; (b) $\mathbb{F}_5$; (c) $\mathbb{F}_{23}$.