ALGEBRA PRELIM—MAY 2016

Work two problems from each of the four sections, i.e., eight in total. Clearly indicate which two problems from each section are to be graded. Otherwise, we will grade problems 1, 2, 4, 5, 7, 8, 10 & 11. In grading, the problems will be weighted equally.

All rings are assumed to have a 1-element.

Groups

Problem 1: The *special linear group* $\mathrm{SL}(2,\mathbb{Z})$ consists of all 2×2 matrices with integer entries and determinant 1, with matrix multiplication as the group operation. Show that $\mathrm{SL}(2,\mathbb{Z})$ is generated by the two elements

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Problem 2: Let G be a group of order $1105 = 5 \cdot 13 \cdot 17$. Show that G is cyclic.

Problem 3: Let $n \geq 1$ be an integer. Show that there is an isomorphism $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \simeq \mathbb{Z}/n\mathbb{Z}$ of abelian groups.

Rings

Problem 4: Let R be a commutative ring, and let I and J be two co-maximal ideals in R, i.e., $I + J = R$. Pick $i \in I$ and $j \in J$ with $i + j = 1$, and define $\varphi\colon I \oplus J \to I \cap J$ by $\varphi(x, y) = jx - iy$. Show that φ is a surjective homomorphism, and that $\ker(\varphi) \simeq R$ (as a module).

Problem 5: Let R be a commutative ring. Recall that an element $x \in R$ is *idempotent* if $x = x^2$, and *nilpotent* if $x^n = 0$ for some $n \geq 1$.

(a) Show that 0 is the only element that is both nilpotent and idempotent.
(b) Show that if x is idempotent, then so is $1 - x$.
(c) Show that if x is nilpotent, then $1 + x$ is a unit.

Problem 6: Let $\mathrm{M}_n(\mathbb{R})$ denote the ring of $n \times n$ matrices over $\mathbb{R}$. Show that the only ideals in $\mathrm{M}_n(\mathbb{R})$ are (0) and $\mathrm{M}_n(\mathbb{R})$.

Modules

Problem 7: Let R be a commutative ring, and let M be an R-module. Consider the map $\delta\colon M \to \mathrm{Hom}_R(\mathrm{Hom}_R(M, R), R)$ given by $\delta(m)(\varphi) = \varphi(m)$.

(a) Show that δ is a homomorphism of R-modules.

(b) Assume that R is a domain and show that $\ker(\delta) = \{m \in M \mid rm = 0 \text{ for some } r \in R \setminus (0)\}$.

Problem 8: Let R be a commutative ring. An R-module M is called *cyclic* if $M = Rm$ for some $m \in M$. Show that if M is cyclic, then there exists an ideal $I \subseteq R$ and an isomorphism $M \simeq R/I$.

Problem 9: Let R be a ring. Show that a left R-module P is projective if and only if there exists a free left R-module F with $F \simeq P \oplus F$.

Fields

Problem 10: Show that $\mathbb{Q}(\sqrt{3 + 2\sqrt{2}})/\mathbb{Q}$ is a Galois extension and determine the Galois group.

Problem 11: Let M/K be a Galois extension of degree n. Show that $M \otimes_K M \simeq M^n$ as rings, when M^n is considered a ring by coordinate-wise addition and multiplication.

Problem 12: Consider the field extension $M/K = \mathbb{F}_2(\sqrt{x}, \sqrt{y})/\mathbb{F}_2(x, y)$, where $\mathbb{F}_2$ is the field with two elements, and x and y are indeterminates. Show that there are infinitely many intermediate fields L, $K \subsetneq L \subsetneq M$.