ALGEBRA PRELIM—AUGUST 2016

Work two problems from each of the four sections, i.e., eight in total. Clearly indicate which two problems from each section are to be graded. Otherwise, we will grade problems 1, 2, 4, 5, 7, 8, 10 & 11. In grading, the problems will be weighted equally.

All rings are assumed to have a 1-element.

Groups

Problem 1: Let φ be an automorphism on the ring $\mathbb{Q}[\![x]\!]$ of formal power series over $\mathbb{Q}$. Show that $\varphi(x) = ux$ for a unit u in $\mathbb{Q}[\![x]\!]$. This defines a map $\varphi \mapsto u = \varphi(x)/x$ from the automorphism group $\operatorname{Aut}(\mathbb{Q}[\![x]\!])$ to the group $\mathbb{Q}[\![x]\!]^*$ of units in $\mathbb{Q}[\![x]\!]$. Is this map an isomorphism?

Problem 2: Let D_m denote the dihedral group of order $2m$. Show: If $n > 2$ is odd, then $D_{2n} \simeq D_n \times \mathbb{Z}/2$.

Problem 3: Let G be a group of order $595 = 5 \cdot 7 \cdot 17$. Show that G is cyclic.

Rings

Problem 4: Let $\mathbb{F}_q$ be the finite field with $q = p^n$ elements, where p is the characteristic. Let $\pi \in \mathbb{F}_q[t]$ be an irreducible polynomial, and let $e \geq 1$. Show that if f is an invertible element in $R = \mathbb{F}_q[t]/\langle \pi^e \rangle$ then the order of f is dp^t for some $d \mid q^{\deg \pi} - 1$ and some $t < e$. [Hint: Induction on e.]

Problem 5: Let R be a commutative ring. Recall that an element $x \in R$ is *nilpotent* if $x^n = 0$ for some $n \geq 1$.

(a) Show that the set $\operatorname{nil}(R)$ of all nilpotent elements in R is an ideal.
(b) Show that $\operatorname{nil}(R)$ is contained in every prime ideal of R.
(c) Show that $\operatorname{nil}(R)$ is a maximal ideal if and only if every element of R is either a unit or nilpotent.

Problem 6: Let R be a commutative ring, and let x and y be indeterminates. Show that $R[x] \otimes_R R[y] \simeq R[x, y]$ as rings.

Modules

Problem 7: Let R be a commutative ring, and consider an exact sequence

$$\eta\colon 0 \to K \to L \to M \to 0$$

of R-modules. η is said to be *pure* if $N \otimes_R \eta$ is exact for every R-module N, and *split* if $\operatorname{Hom}_R(N, \eta)$ is exact for every R-module N.

(a) Show that if η is pure and I is an injective R-module, then the exact sequence $\operatorname{Hom}_R(\eta, I)$ is split.

(b) Show that if I is a faithfully injective R-module and the exact sequence $\operatorname{Hom}_R(\eta, I)$ is split, then η is pure.

Problem 8: Let R be a commutative ring. An R-module M is called *simple* if $M \neq 0$ and its only submodules are 0 and M. Show that if M is simple, then there exists a maximal ideal $I \subsetneq R$ and an isomorphism $M \simeq R/I$.

Problem 9: Let $\mathrm{M}_n(\mathbb{R})$ denote the ring of $n \times n$ matrices over $\mathbb{R}$. Show that any left $\mathrm{M}_n(\mathbb{R})$-module is a direct sum of submodules isomorphic to $\mathbb{R}^n$.

Fields

Problem 10: Let M be the splitting field over $\mathbb{Q}$ of $x^3 - 2x + 2$. Determine the Galois group of $M/\mathbb{Q}$.

Problem 11: Let M/K be a Galois extension with Galois group G. The *skew group algebra* $M\{G\}$ then consists of formal linear combinations $\sum_{\sigma \in G} a_\sigma \sigma$, where $a_\sigma \in M$, with addition given term-by-term, and multiplication given by $(a\sigma)(b\tau) = (a\sigma(b))\sigma\tau$. $M\{G\}$ is then an associative ring with identity. (You do not need to show that.) Show: $j\colon M\{G\} \to \operatorname{End}_K(M)$, given by $j\big(\sum_{\sigma \in G} a_\sigma \sigma\big)\colon x \mapsto \sum_{\sigma \in G} a_\sigma \sigma(x)$, is a ring isomorphism.

Problem 12: Let K be a field of prime characteristic p, and let α be algebraic over K. Show that α is separable over K if and only if $K(\alpha) = K(\alpha^p)$.