ALGEBRA PRELIM—MAY 2018

Work two problems from each of the four sections, i.e., eight in total. Clearly indicate which two problems from each section are to be graded. Otherwise, we will grade problems 1, 2, 4, 5, 7, 8, 10, and 11. In grading, the problems will be weighted equally.

All rings are assumed to have a 1-element, and ring homomorphisms preserve 1.

Groups

Problem 1: Let A be a subgroup of $\mathbb{Z}^n$ for some natural number n, and let $\varphi\colon A \to \mathbb{Q}$ be a group homomorphism. Show that φ extends to $\mathbb{Z}^n$, i.e., that there exists a homomorphism $\Phi\colon \mathbb{Z}^n \to \mathbb{Q}$ with $\Phi(\mathbf{x}) = \varphi(\mathbf{x})$ for $\mathbf{x} \in A$.

Problem 2: Let G be a group and let $x, y \in G$. Recall that the *commutator* of x and y is $[x, y] := x^{-1}y^{-1}xy$, and the *commutator subgroup* of G is $G' := \langle\, [x, y] \mid x, y, \in G \,\rangle$. For a p-group P, prove that P is cyclic if and only if P/P' is cyclic.

Problem 3: Let G be a group of order $255 = 3 \cdot 5 \cdot 17$. Show that G is cyclic.

Rings

Problem 4: Let R be a ring and $f(x)$ an element of the polynomial ring $R[x]$. Prove that $f(x)$ is a zero divisor in $R[x]$ if and only if there exists $0 \neq b \in R$ such that $bf(x) = 0$.

Problem 5: Let $\varphi\colon R \to S$ be a homomorphism of commutative rings.

(a) Prove that if P is a prime ideal of S then the preimage $\varphi^{-1}(P)$ is a prime ideal of R.
(b) Apply part (a) to the special case when R is a subring of S and φ is the inclusion homomorphism to deduce that if P is a prime ideal of S, then $P \cap R$ is a prime ideal of R.

(c) Prove that if M is a maximal ideal of S and φ is surjective then $\varphi^{-1}(M)$ is a maximal ideal of R. Give an example to show that this need not be the case if φ is not surjective.

Problem 6: Let R be a UFD. Suppose that for every pair of relatively prime non-units $p, q \in R$ the ideal $Rp + Rq$ is principal. Prove that for every pair of elements $a, b \in R$ the ideal $Ra + Rb$ is principal.

Modules

Problem 7: Let R be a commutative ring and $I \subset R$ be a finitely generated ideal. An R-module M is called *I-torsion* if every element of M is annihilated by some power of I. That is, for every $m \in M$ there is a natural number n such that $xm = 0$ for every $x \in I^n$. Show that in an exact sequence

$$0 \to M' \to M \to M'' \to 0$$

the module M is I-torsion if and only if M' and M'' are I-torsion.

Problem 8: Let k be field. Let $k[x, y]$ be the ring of polynomials in two indeterminates x, y over k. Show that

$$k[x, y]/(y - x) \qquad \text{and} \qquad k[x, y]/(y - 1)$$

are isomorphic as $k[x]$-modules, but are not isomorphic as $k[x, y]$-modules.

Problem 9: Show that the field of fractions F of an integral domain D is both an injective and a flat D-module.

Fields

Problem 10: Show that there exists a field extension L/K of degree 4, such that $L = K(\alpha)$ for all $\alpha \in L \setminus K$.

Problem 11: Let a be an integer, and set $p(x) = x^3 + ax^2 - (a+3)x + 1$. Show that $p(x)$ is irreducible over $\mathbb{Q}$. Then show: If α is a root of $p(x)$, then so is $1/(1 - \alpha)$. Use this to determine the Galois group of the splitting field over $\mathbb{Q}$

Problem 12: Let p be an odd prime, and let ζ be a primitive p^{th} root of unity. Show that $(-1)^{(p-1)/2}\, p$ is a square in $\mathbb{Q}(\zeta)$. [Hint: Use that $\Phi_p(1) = p$, where $\Phi_p(x)$ is the p-th cyclotomic polynomial.]