ALGEBRA PRELIM—AUGUST 2018

Work two problems from each of the four sections, i.e., eight in total. Clearly indicate which two problems from each section are to be graded. Otherwise, we will grade problems 1, 2, 4, 5, 7, 8, 10, and 11. In grading, the problems will be weighted equally.

All rings are assumed to have a 1-element, and ring homomorphisms preserve 1.

Groups

Problem 1: Let G be a finite group of order n. Assume that there is *at most* one subgroup of G of order d for each $d \mid n$. Show that G is cyclic.

Problem 2: Recall that the *outer automorphism group* for a group G is the quotient group

$$\mathrm{Out}(G) = \mathrm{Aut}(G)/\,\mathrm{Inn}(G).$$

Show that $\mathrm{Out}(G)$ acts in a natural way as automorphisms on $Z(G)$.

Problem 3: Prove that there is no simple group of order 180.

Rings

Problem 4: Let $k[x, y]$ be the polynomial ring in two indeterminates x, y over a field k. Show that $k[x, y]$ is a unique factorization domain but not a principal ideal domain.

Problem 5: Let $\varphi\colon R \to S$ be a homomorphism of rings. In the following, I denotes an ideal in R and J denotes an ideal in S.

(1) Show that the preimage $\varphi^{-1}(J) \subset R$ is an ideal.
(2) Show by example that the image $\varphi(I)$ need not be an ideal in S.
(3) Denote by $\overline{\varphi(I)}$ the ideal in S generated by the set $\varphi(I)$. Show that there are containments of ideals

$$I \subseteq \varphi^{-1}(\overline{\varphi(I)}) \qquad \text{and} \qquad \overline{\varphi(\varphi^{-1}(J))} \subseteq J\,;$$

and show by example that equality need not hold.
(4) Show that there are equalities of ideals

$$\overline{\varphi(I)} = \varphi(\varphi^{-1}(\overline{\varphi(I)}) \qquad \text{and} \qquad \varphi^{-1}(\overline{\varphi(\varphi^{-1}(J))}) = \varphi^{-1}(J)\,;$$

Problem 6: Let $\mathfrak{p}$ be a prime ideal in a commutative ring R and set $S = R \setminus \mathfrak{p}$.

(a) Show that S is a multiplicative subset of R. (Recall a subset S of R is called a multiplicative set if it is closed under multiplication and does not contain the zero element nor any zero divisors.)

(b) Show that $S^{-1}R\,\mathfrak{p}$ is the unique maximal ideal in $S^{-1}R$.

Modules

Problem 7: Let R be a commutative ring and M be an R-module. A submodule $N \subseteq M$ is said to be *small* if $N + M' \neq M$ for every submodule $M' \neq M$ of M.

(a) Consider R-(sub)modules $N' \subseteq N \subseteq M$. Show that if N' is small in N, and N is small in M, then N' is small in M.

(b) Show that if N_1 and N_2 are small submodules of M, then $N_1 + N_2$ is a small submodule of M.

Problem 8: Let M and N be modules over a commutative ring R. Show that for elements $m_1, \ldots, m_p$ in M and $n_1, \ldots, n_p$ in N with $\sum_{i=1}^{p} m_i \otimes n_i = 0$ in $M \otimes_R N$, there exist finitely generated submodules $M' \subseteq M$ and $N' \subseteq N$ with $m'_1, \ldots, m'_p \in M'$ and $n'_1, \ldots, n'_p \in N'$ such that $\sum_{i=1}^{p} m'_i \otimes n'_i = 0$ holds in $M' \otimes_R N'$.

Problem 9: Let R be a ring and let e be a non-zero element of R satisfying $e^2 = e$. Prove that the R-module Re is projective. (Note that R is not necessarily an integral domain).

Fields

Problem 10: Let $p(x)$ be an irreducible polynomial in $\mathbb{Q}[x]$ of degree 4, with exactly two real roots, and let M be the splitting field over $\mathbb{Q}$. Show that $\mathrm{Gal}(M/\mathbb{Q})$ cannot be isomorphic to the Klein group V_4 or the alternating group A_4.

Problem 11: Show that an element $\alpha \in \mathbb{Q}(i)$ has a minimal polynomial (over $\mathbb{Q}$) with integer coefficients if and only if the real and imaginary parts of α are both integers.

Problem 12: Let p be a prime number. Show that $\mathbb{Q}(\sqrt[4]{p}, i)/\mathbb{Q}$ is a Galois extension and determine the Galois group.