ALGEBRA PRELIM—MAY 2019

Work two problems from each of the four sections, i.e. eight in total. Clearly indicate which two problems from each section are to be graded; otherwise, problems 1, 2, 4, 5, 7, 8, 10, and 11 will be graded. In grading, the problems will be weighted equally.

All rings are assumed to have a 1-element, and ring homomorphisms preserve 1.

Groups

Problem 1: Let G be a group and $H \subseteq G$ a subgroup of finite index. Show that G contains a normal subgroup N of finite index, such that $N \subseteq H$.

Problem 2: A *maximal* subgroup of a group G is a proper subgroup H such that there are no subgroups K with $H \subsetneq K \subsetneq G$. For a non-trivial finite group G, the *Frattini subgroup* $\Phi(G)$ is the intersection of all the maximal subgroups. Show that an element $g \in G$ is in the Frattini subgroup if and only if it satisfies the following condition: If the group elements $g, g_1, \ldots, g_k$ generate G, then $g_1, \ldots, g_k$ generate G. (I.e. $\Phi(G)$ consists of those elements that are never needed in a generating set for G.)

Problem 3: Prove that there are no simple groups of order $4389 = 3 \cdot 7 \cdot 11 \cdot 19$.

Rings

Problem 4: Let R be a commutative ring that satisfies the *descending chain condition,* that is, every descending chain of ideals

$$I_1 \supseteq I_2 \supseteq \cdots \supseteq I_i \supseteq I_{i+1} \supseteq \cdots$$

is eventually stationary. (There is an $N \in \mathbb{N}$ such that $I_n = I_{n+1}$ holds for all $n \geq N$.) Prove that R has only finitely many maximal ideals.

Problem 5: Let R be a ring and e an idempotent in R; that is, $e^2 = e$.

(a) Prove that the set

$$eRe = \{ere \mid r \in R\}$$

under the operations inherited from R is a ring with a 1-element.

(b) Show that for every ideal $I \subseteq R$ the set $\{ere \mid r \in I\}$ is an ideal in eRe.

Problem 6: Prove that every prime ideal in a finite commutative ring is maximal.

Modules

Problem 7: In the product ring $R = \times_{n\in\mathbb{N}} \mathbb{Q}$ (countably many copies of $\mathbb{Q}$) consider the ideals

$$I_n = \{(q_i)_{i\in\mathbb{N}} \in R \mid q_i = 0 \text{ for all } i \neq n\}\,.$$

(a) Show that each I_n is an injective R-module.
(b) Show that the direct sum $\bigoplus_{n\in\mathbb{N}} I_n$ is not an injective R-module.

Problem 8: Let R be a ring and M an R-module. A submodule $E \subseteq M$ is called *essential* if $E \cap M' \neq 0$ holds for every submodule $M' \neq 0$ of M.

(a) Show that if E' and E'' are essential submodules of M, then $E' \cap E''$ is an essential submodule of M.
(b) Show that if E is an essential submodule of M and E' is an essential submodule of E, then E' is an essential submodule of M.

Problem 9: Let A be a complex 3×3 matrix. Decide if the following information is sufficient to determine the Jordan Canonical Form of A:

- The characteristic polynomial of A is $(x-1)^3$.
- $(A-I)^2 = 0$.
- $A \neq I$.

Fields

Problem 10: A polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ of degree n is called *symmetric* if $a_0 = a_n,\ a_1 = a_{n-1},\ \ldots,\ a_i = a_{n-i},\ \ldots$ Show that for every natural number $n > 1$, the cyclotomic polynomial $\Phi_n(x)$ is symmetric.

Problem 11: The polynomial

$$f(x) = x^6 + 10x^5 + 30x^4 + 20x^3 - 45x^2 - 72x - 27$$

is irreducible in $\mathbb{Q}[x]$. (You do not need to show that.) Show that if α is a root of $f(x)$, then so is $-3/(\alpha+3)$. Use this to determine the Galois group over $\mathbb{Q}$ for the splitting field of $f(x)$.

Problem 12: Let $p \geq 3$ be a prime. How many elements of the field $\mathbb{F}_p$ have a square root?