# ALGEBRA PRELIM—AUGUST 2020

Work two problems from each of the four sections, i.e. eight in total. Clearly indicate which two problems from each section are to be graded; otherwise, problems 1, 2, 4, 5, 7, 8, 10, and 11 will be graded. In grading, the problems will be weighted equally.

All rings are assumed to have 1-elements, and all ring homomorphisms are assumed to preserve 1-elements.

## Groups

**Problem 1:** Let $G$ be a finite group.

(a) Prove: If $H$ is a subgroup of $G$, then the number of non-identity elements of $G$ that are contained in conjugates of $H$ is at most

$$(|H|-1)\,[G:H]$$

(b) A *maximal* subgroup of $G$ is a proper subgroup $H$ such that there are no subgroups $K$ with $H \subsetneq K \subsetneq G$. Assume that $G$ is simple, and that every proper subgroup of $G$ is abelian. Show: If $H_1$ and $H_2$ are distinct maximal subgroups of $G$, then $H_1 \cap H_2 = \{1\}$.

**Problem 2:** Let $G$ be a finite group, and let $p$ be a prime number. Show that there exists a unique smallest normal subgroup of $G$ with index relatively prime to $p$.

**Problem 3:** Prove that there are no simple groups of order 144.

## Rings

**Problem 4:** Let $\mathrm{M}_2(\mathbb{Z})$ be the ring of $2 \times 2$-matrices over the integers. Set

$$J = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad R = \{A \in \mathrm{M}_2(\mathbb{Z}) \mid AJ = JA\}\,.$$

(a) Show that $R$ is a subring of $\mathrm{M}_2(\mathbb{Z})$.
(b) Show that $R$ is commutative.

**Problem 5:** Let $R$ be a ring, and let $C$ be the left ideal in $R$ generated by the set

$$\{xy - yx \mid x, y \in R\}$$

This is the *commutator ideal* in $R$.

(a) Prove that $C$ is also a right ideal in $R$.
(b) Prove that the quotient ring $R/C$ is commutative.

**Problem 6:** Let $I$ be a non-zero ideal in $\mathbb{Z}[x]$, and assume that $I \cap \mathbb{Z} = \{0\}$. Show that there exists a non-constant polynomial $f(x) \in \mathbb{Z}[x]$ that divides all elements in $I$. Does this mean that $I$ is a principal ideal?

## Modules

**Problem 7:** Let $\varphi\colon R \to S$ be a ring homomorphism, and let $M$ be a left $R$-module. We consider $S$ to be a left $R$-module through $\varphi$.

(a) Show that $\operatorname{Hom}_R(S, M)$ is a left $S$-module with scalar multiplication given by
$$(s \cdot f)(t) = f(ts)$$
for $f \in \operatorname{Hom}_R(S, M)$ and $s, t \in S$.

(b) Show: If $M$ is an injective $R$-module, then $\operatorname{Hom}_R(S, M)$ is an injective $S$-module.

**Problem 8:** Show that the matrix $A = \left(\begin{smallmatrix} 0 & -5 \\ 1 & -2 \end{smallmatrix}\right)$ satisfies $A^2 + 2A + 5I = 0$. Use this to prove that there is a real $n \times n$ matrix with minimal polynomial $t^2 + 2t + 5$ if and only if $n$ is even.

**Problem 9:** Let $R$ be a commutative ring. For an $R$-module $M$, we define the *generator ideal* for $M$ as
$$I_R(M) = \sum_{f \in \operatorname{Hom}_R(M,R)} f(M)$$

(a) Show that $I_R(M)$ is an ideal in $R$.

(b) Assume that $I_R(M) = R$. Show: If $N$ is a non-trivial $R$-module, then $M \otimes_R N \neq \{0\}$.

## Fields

**Problem 10:** Let $M$ be the splitting field over $\mathbb{Q}$ of the polynomial

$$x^3 - 3x + 3$$

What is the Galois group for $M/\mathbb{Q}$?

**Problem 11:** Let $F$ be a field, and let $K \subseteq F$ be a subfield containing all the squares in $F$, i.e., all elements of the form $\alpha^2$ for $\alpha \in F$.

(a) Show: If $F$ does not have characteristic 2, then $K = F$.
(b) Show: If $F$ is finite of characteristic 2, then $K = F$.

**Problem 12:** Consider the polynomial

$$f(x) = x^4 + ax^3 - 6x^2 - ax + 1$$

(a) Show that $f(x)$ is irreducible in $\mathbb{Q}[x]$ for all $a \in \mathbb{Z}$, except $a = 0$ and $a = \pm 3$.
(b) Show: If $\alpha$ is a root of $f(x)$, then $(1+\alpha)/(1-\alpha)$ is a root as well.
(c) Use this information to determine the Galois group over $\mathbb{Q}$ for the splitting field of $f(x)$ for $a \in \mathbb{Z}$, $a \neq 0, \pm 3$.