ALGEBRA PRELIM—AUGUST 2022

Work two problems from each of the four sections, i.e., eight in total. Clearly indicate which two problems from each section are to be graded. Otherwise, we will grade problems 1, 2, 4, 5, 7, 8, 10 & 11. In grading, the problems will be weighted equally.

All rings are assumed to have 1-elements, and all ring homomorphisms are assumed to preserve the 1-elements.

Groups

Problem 1: (a) Let A, B and C be finitely generated Abelian groups, and assume that $A \times B \simeq A \times C$. Show that $B \simeq C$.

(b) Give an example of Abelian groups A, B and C, where $A \times B \simeq A \times C$ but $B \not\simeq C$.

Problem 2: Let G be a finite Abelian group of order n. Show that G is isomorphic to a transitive subgroup of S_m if and only if $m = n$.

Problem 3: A *Carmichael number* is a composite natural number n such that

$$\forall a \in \mathbb{Z} \colon a^n \equiv a \pmod{n}$$

The smallest Carmichael number is $561 = 3 \cdot 11 \cdot 17$. Let n be a Carmichael number. Show that any Abelian group of order n is cyclic.

Rings

Problem 4: Let R be a non-trivial commutative ring. Recall that an element $x \in R$ is called *idempotent* if $x^2 = x$.

(a) Show that 1 and 0 are the only idempotents in an integral domain.

(b) Show that if every element of R is idempotent, then R has characteristic 2.

(c) Show that if every element of R is idempotent, then R is commutative.

Problem 5: Let $p(x)$ and $q(x)$ be primitive polynomials in $\mathbb{Z}[x]$. Show that $p(x)$ and $q(x)$ have greatest common divisor 1 if and only if there exist polynomials $f(x)$ and $g(x)$ in $\mathbb{Z}[x]$ such that $f(x)p(x) + g(x)q(x)$ is a non-zero constant.

Problem 6: The ring of *Hurwitz integers* is the subring

$$H = \left\{ \frac{a + bi + cj + dk}{2} \,\middle|\, a, b, c, d \in \mathbb{Z},\ a \equiv b \equiv c \equiv d \pmod 2 \right\}$$

of the quaternions. (You may take it for granted that it *is* a subring.) Recall that the *norm* of a quaternion $x = s + ti + uj + vk$ is $N(x) = s^2 + t^2 + u^2 + v^2$, i.e., the determinant of x, when x is considered as a 2×2 complex matrix.

(a) Show that $N(x)$ is an integer when $x \in H$.

(b) Show that H allows *right division with remainder* with respect to the norm: For $x, y \in H$, $y \neq 0$, there exists $q, r \in H$ with

$$x = qy + r, \quad N(r) < N(y)$$

(c) Show that all left ideals in H have the form

$$Hx = \{qx \mid q \in H\}$$

for some $x \in H$.

Modules

Problem 7: Let R be a ring, and consider the following commutative diagram of R-module homomorphisms:

$$\begin{array}{ccccccc} M' & \xrightarrow{\alpha'} & M & \xrightarrow{\alpha} & M'' & \longrightarrow & 0 \\ {\scriptstyle \psi'}\downarrow & & \downarrow{\scriptstyle \psi} & & & & \\ N' & \xrightarrow[\beta']{} & N & \xrightarrow[\beta]{} & N'' & & \end{array}$$

Assume that the upper row is exact and that $\beta\beta' = 0$. Show that there exists a unique homomorphism $\psi''\colon M'' \to N''$ with $\psi''\alpha = \beta\psi$.

Problem 8: Let R be an integral domain, and let I be an ideal in R.

(a) Show that an R-module homomorphism $\varphi\colon I \to R$ has the form

$$\varphi(x) = ax$$

for some a in the field of fractions for R.

(b) Assume that I is projective as an R-module. Show that I is finitely generated.

Problem 9: Let P be a projective left S-module and M an R-S-bimodule. Assume that M is projective as an R-module. Show that $M \otimes_S P$ is projective as an R-module as well.

Fields

Problem 10: Let M be the splitting field for the polynomial

$$x^4 - 10x^2 + 5$$

over $\mathbb{Q}$. Determine the Galois group for $M/\mathbb{Q}$.

Problem 11: Let M/F be a Galois extension with $\mathrm{Gal}(M/F) = GH$, where G and H are normal subgroups of $\mathrm{Gal}(M/F)$ with $G \cap H = \{1\}$. Show that

$$a \otimes b \mapsto ab$$

is an isomorphism $M^G \otimes_F M^H \simeq M$.

Problem 12: Let p be a prime number, and let A be a square matrix over a field of characteristic p. Show that

$$\mathrm{Tr}(A^p) = \mathrm{Tr}(A)^p$$