ALGEBRA PRELIM—MAY 2022

Work two problems from each of the four sections, i.e., eight in total. Clearly indicate which two problems from each section are to be graded. Otherwise, we will grade problems 1, 2, 4, 5, 7, 8, 10 & 11. In grading, the problems will be weighted equally.

All rings are assumed to have 1-elements, and all ring homomorphisms are assumed to preserve the 1-elements.

Groups

Problem 1: Let G be a group, and let H be a subgroup of finite index $n = [G : H]$. Show that

$$\forall g \in G \, \exists i \in \{1, 2 \ldots, n\} \colon g^i \in H.$$

Problem 2: Let p be a prime, and let $G = S_p$ be the symmetric group on p letters. Show that, if a subgroup $H \leq G$ has order p, then there is an isomorphism

$$\mathrm{N}_G(H)/\mathrm{C}_G(H) \simeq \mathrm{Aut}(H).$$

Problem 3: Determine, up to isomorphism, all groups of order 75.

Rings

Problem 4: Let n be a natural number, and consider $\mathbb{Q}^n$ as a $\mathbb{Q}$-algebra with coordinate-wise addition and multiplication. Show that there is a bijective correspondence between algebra homomorphisms $\varphi\colon \mathbb{Q}^n \to \mathbb{Q}^n \otimes_{\mathbb{Q}} \mathbb{Q}^n$ and binary operations $*$ on the set $\{1, 2, \ldots, n\}$, given by

$$\varphi(\mathbf{e}_k) = \sum_{i*j=k} \mathbf{e}_i \otimes \mathbf{e}_j, \quad 1 \le k \le n,$$

where $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are the standard basis vectors.

Problem 5: Let R be a commutative ring and M an R-module.

(a) Show that $R \times M$, with coordinate-wise addition, and with multiplication given by

$$(r, m)(r', m') = (rr', rm' + r'm),$$

is a commutative ring.

(b) Show that $\{0\} \times M$ is an ideal in this ring.

Problem 6: Let R be a commutative ring. If, for every $x \in R$, there exists $y \in R$ with $xyx = x$, then R is said to be *von Neumann regular*.

(a) Show that every field is von Neumann regular.

(b) Show that a von Neumann regular integral domain is a field.

(c) Recall that an element $x \in R$ is called *idempotent* if $x^2 = x$. Show that if R is von Neumann regular, then every principal ideal of R is generated by an idempotent.
[Hint: If $xyx = x$ holds, then x and xy generate the same ideal.]

Modules

Problem 7: The matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

has characteristic polynomial $x^4 + x^3 + x^2 + x + 1$. We define a $\mathbb{Q}[x]$-module structure on $\mathbb{Q}^4$ by

$$f(x) \cdot \mathbf{u} = f(A)\mathbf{u}$$

for $f(x) \in \mathbb{Q}[x]$ and $\mathbf{u} \in \mathbb{Q}^4$, i.e., by letting x act as multiplication by A. Show that $\mathbb{Q}^4$ is a cyclic $\mathbb{Q}[x]$-module, and that in fact *every* non-zero vector $\mathbf{u} \in \mathbb{Q}^4$ is a generator.

Problem 8: Let R be a ring, and consider the following commutative diagram of R-module homomorphisms:

$$\begin{tikzcd} & M' \arrow[r, "\alpha'"] & M \arrow[r, "\alpha"] \arrow[d, "\varphi"'] & M'' \arrow[d, "\varphi''"] \\ 0 \arrow[r] & N' \arrow[r, "\beta'"'] & N \arrow[r, "\beta"'] & N'' \end{tikzcd}$$

Assume that the lower row is exact and that $\alpha\alpha' = 0$. Show that there exists a unique homomorphism $\varphi'\colon M' \to N'$ with $\beta'\varphi' = \varphi\alpha'$.

Problem 9: Let R be a commutative ring, and let M be an R-module satisfying the *ascending chain condition*, i.e., every ascending chain

$$N_1 \subseteq N_2 \subseteq \cdots \subseteq N_i \subseteq N_{i+1} \subseteq \cdots$$

of submodules is eventually stationary: $N_n = N_{n+1}$ for all sufficiently large n.

(a) Show that M is generated by a finite number of elements.

(b) Show that every surjective R-module homomorphism $\varphi\colon M \to M$ is an isomorphism. [Hint: Consider iterations of φ.]

Fields

Problem 10: Let $f(x) = x^3+ax^2+bx+c \in F[x]$ be a cubic polynomial over the field F. Show that $f(x)$ has two roots that add up to 0 if and only if $c = ab$.

Problem 11: Let F be a field, and let M be the splitting field over F for some monic polynomial in $F[x]$. Let $G = \mathrm{Aut}_F(M)$ be the group of F-automorphisms on M, and let $K = M^G$ be the fixed field for G. Show that, for any element $\alpha \in K$, the minimal polynomial for α over F has α as its only root. [Note: This is *not* the same as saying that it has degree 1.]

Problem 12: Let F be a field, and let $f(x) \in F[x]$ be an irreducible polynomial of prime degree p. Let θ be a root of $f(x)$, and assume that the root field $F(\theta)$ contains more than one root of $f(x)$. Show that $f(x)$ is separable, and that $F(\theta)$ is the splitting field for $f(x)$ over F.