ALGEBRA PRELIM—AUGUST 2023

Work two problems from each of the four sections, i.e., eight in total. Clearly indicate which two problems from each section are to be graded. Otherwise, we will grade the first two problems in the section. In grading, the problems will be weighted equally.

All rings are assumed to have 1-elements, and all ring homomorphisms are assumed to preserve the 1-elements.

Groups

Problem 1: Let G be a finite group, and let N be a normal subgroup. Assume that $\gcd(|N|, [G : N]) = 1$. Show that N is the only subgroup of G of order $|N|$.

Problem 2: Let G be a finite group, and let p be the smallest prime divisor of $|G|$. Assume that p is odd and divides $|G|$ exactly twice. Show that a Sylow p-subgroup of G is normal if and only if it is contained in $Z(G)$.

Problem 3: Let G be a group. Show that the following three conditions are equivalent for a subgroup H of $G \times G$:

(1) $\forall g \in G\colon (g, g) \in H$.

(2) There exists a normal subgroup $N \triangleleft G$, such that

$$H = \bigcup_{g \in G} (gN \times gN)$$

(3) The relation $\sim$ on G, given by

$$g \sim g' \Leftrightarrow (g, g') \in H,$$

is an equivalence relation.

Rings

Problem 4: Let p be a prime, and let ζ be a primitive p^{th} root of unity. Show that $\zeta \mapsto [1]$ defines a ring homomorphism $\mathbb{Z}[\zeta] \to \mathbb{Z}/p\mathbb{Z}$, and that the kernel is the principal ideal generated by $\zeta - 1$.

Problem 5: Let R be a finite-dimensional commutative algebra over a field F. Show that R is isomorphic to a product of finite field extensions of F if and only if the intersection of all the maximal ideals in R is $\{0\}$.

Problem 6: Let R be an integral domain in which every prime ideal is principal. Show that R is a principal ideal domain.

Modules

Problem 7: Let M and N be S-R-bimodules.

(a) Show that $\mathrm{Hom}_R(M, N)$ has an S-S-bimodule structure, with the left S-module structure given by $(s\varphi)(m) = s\varphi(m)$ and the right S-module structure by $(\varphi s)(m) = \varphi(sm)$.

(b) Show that if $R = S$ is commutative and the left and right R-module structures on M and N are the same, then they are the same on $\mathrm{Hom}_R(M, N)$ as well, i.e., $s\varphi$ is always equal to φs.

Problem 8: Let R be a commutative ring, and let $\varphi\colon R^n \to R^n$ be a module homomorphism. Assume that φ is onto. Show that φ is then also one-to-one.

Problem 9: Let M be a non-trivial finitely generated torsion-free module over an integral domain R. Show that there exists a non-zero R-module homomorphism $M \to R$.

Fields

Problem 10: Let p be a prime, and let $\mathbb{F}_p$ be the field with p elements. Let $L = \mathbb{F}_p(x, y)$ and $K = \mathbb{F}_p(x^p, y^p)$, where x and y are indeterminates.

(a) Determine $[L : K]$.
(b) Show that L/K is not a simple extension.

Problem 11: Consider the polynomial

$$p(x) = x^4 + 5x^3 + 10x^2 + 10x + 5 \in \mathbb{Q}[x],$$

and let M be its splitting field over $\mathbb{Q}$. Show that if θ is a root of $p(x)$, then so is $\theta^2 + 2\theta$. Use this to determine $\mathrm{Gal}(M/\mathbb{Q})$.

Problem 12: Let M/F be a Galois extension, and let L/F be a subextension. Show that $M \otimes_F L$ (as a ring) is isomorphic to $M^{[L:F]}$ (with coordinate-wise addition and multiplication).