ALGEBRA PRELIM—MAY 2023

Work two problems from each of the four sections, i.e., eight in total. Clearly indicate which two problems from each section are to be graded. Otherwise, we will grade the first two problems in the section. In grading, the problems will be weighted equally.

All rings are assumed to have 1-elements, and all ring homomorphisms are assumed to preserve the 1-elements.

Groups

Problem 1: Let $k \in \mathbb{N}$ be such that $p = 6k + 1$, $q = 12k + 1$ and $r = 18k + 1$ are all primes. (The first few such k's are 1, 6, 35 and 45, and there are many others. Conjecturally, there are infinitely many.) Show that a group of order pqr is cyclic.

Problem 2: For a field F, we define the *special linear group* $\mathrm{SL}(n, F)$ to be the group of $n \times n$ matrices over F with determinant 1.

(a) Determine $|\,\mathrm{SL}(2, \mathbb{F}_4)|$, where $\mathbb{F}_4$ is the field with four elements.
(b) Show that $\mathrm{SL}(2, \mathbb{F}_4)$ is a simple group.

Problem 3: Let G be a group, and let H be a subgroup of finite index n. Show that $g^n \in H$ for all $g \in Z(G)$.

Rings

Problem 4: Let R be a unique factorization domain with field of fractions $F = Q(R)$. Show that the multiplicative group F^* for F is the direct product of the multiplicative group R^* for R and a free Abelian group. (Note that the free Abelian group is not necessarily finitely generated.)

Problem 5: Let R be a finite-dimensional commutative algebra over a field F. Show that there are only finitely many prime ideals in R, and that they are all maximal.

Problem 6: Let R be a principal ideal domain with field of fractions $F = Q(R)$, let S be a subring of F with $R \subseteq S$, and let

$$D = \{d \in R \mid 1/d \in S\}.$$

Show that

$$S = \{r/d \mid r \in R,\ d \in D\}.$$

Modules

Problem 7: Let R be a commutative ring, and let $\mathrm{M}_n(R)$ the ring of $n \times n$ matrices over R. Also, let $R^{1\times n}$ be the module of $1 \times n$ matrices, and let $\mathbf{e}_i$ be the i^{th} standard basis vector for R^n.

(a) Show that $\mathbf{e}_i^t \otimes \mathbf{e}_i = \mathbf{e}_j^t \otimes \mathbf{e}_j$ in $R^{1\times n} \otimes_{\mathrm{M}_n(R)} R^n$ for all i and j, and that $\mathbf{e}_i^t \otimes \mathbf{e}_j = \mathbf{0}$ for $i \neq j$. (Here, ^{t} denotes transposition.)

(b) Show that $\mathbf{x} \otimes \mathbf{y} \mapsto \mathbf{x}\mathbf{y} = x_1 y_1 + \cdots + x_n y_n$ is an isomorphism

$$R^{1\times n} \otimes_{\mathrm{M}_n(R)} R^n \simeq R$$

of R-R-bimodules.

Problem 8: Let R be a commutative ring, and let I and J be co-maximal ideals in R, i.e., $I + J = R$. Show that I and J are projective (as R-modules) if and only if the product ideal IJ is projective.

Problem 9: Let M be a non-trivial Noetherian module over a ring R. A *maximal* submodule of M is a proper submodule $Q \subset M$, such that there are no submodules properly between Q and M, i.e., submodules S with $Q \subset S \subset M$. An element $m \in M$ is a *non-generator* if and only if it is never needed in a generating set for M: Whenever M is generated by a set $m, m_1, \ldots, m_n$, it is already generated by $m_1, \ldots, m_n$.

(a) Show that every proper submodule of M is contained in a maximal submodule.

(b) Let F be the intersection of all the maximal submodules of M. Show that the elements of F are exactly the non-generators for M.

Fields

Problem 10: Let F be a field of characteristic 2, let n be odd, and let A be a symmetric $n \times n$ matrix over F with zeroes in the diagonal. Show that $\det(A) = 0$.

Problem 11: Show that the polynomial

$$f(x) = x^4 - 10x^2 + 20 \in \mathbb{Q}[x]$$

is irreducible over $\mathbb{Q}$. Let M be the splitting field of $f(x)$ over $\mathbb{Q}$. Determine the structure of $\mathrm{Gal}(M/\mathbb{Q})$.

Problem 12: Let N/F be a field extension, and let K/F and M/F be subextensions, with M/F Galois. Show that $M \otimes_F K$ is a field if and only if $K \cap M = F$.