ALGEBRA PRELIM—AUGUST 2024

Work two problems from each of the four sections, i.e., eight in total. Clearly indicate which two problems from each section are to be graded. Otherwise, we will grade problems 1, 2, 4, 5, 7, 8, 10 & 11. In grading, the problems will be weighted equally.

All rings are assumed to have a 1-element and all ring homomorphisms are assumed to preserve the 1-elements.

Groups

Problem 1: Determine, up to isomorphism, all groups of order $63 = 3^2 \cdot 7$.

Problem 2: A subgroup M of a group G is a *maximal* subgroup if $M \neq G$ and there are no subgroups H of G with $M \subsetneq H \subsetneq G$. For a group G, the *Frattini subgroup* of G, denoted $\Phi(G)$, is the intersection of all the maximal subgroups of G. If G has no maximal subgroups, we set $\Phi(G) = G$. For a finite group G, prove that $\Phi(G)$ is nilpotent.
Hint: Use Frattini's argument to show that every Sylow subgroup of $\Phi(G)$ is normal in G.

Problem 3: For a group G, let $\mathcal{K}_G(g)$ be the conjugacy class of g in G. For $n \geq 3$, let S_n be the symmetric group on n letters and let $A_n \leq S_n$ be the alternating group.

(1) Find all finite groups which have exactly two conjugacy classes.
(2) Let $\sigma \in A_n$. Prove that $\mathcal{K}_{A_5}(\sigma) = \mathcal{K}_{S_5}(\sigma)$ if and only if σ commutes with an odd permutation in S_n.

Rings

Problem 4: Let R be a ring and let $S \subseteq R$ be a subset. The *annihilator* of S in R is defined as

$$\mathrm{Ann}(S) = \{r \in R \mid rs = 0 \text{ for all } s \in S\}.$$

(1) Show that $\mathrm{Ann}(S)$ is a left ideal of R.
(2) If S is a left ideal of R, show that $\mathrm{Ann}(S)$ is a two-sided ideal of R.
(3) If $S = \{x\}$ with $x^2 = x$, show that $\mathrm{Ann}(S)$ is a left principal ideal of R.

Problem 5: Let R be a commutative ring and let $x \in R$. Show that x is an element of every maximal ideal of R if and only if $1 - xy$ is a unit for all $y \in R$.

Problem 6: Let X be a non-empty set. Let $\mathcal{P}(X)$ be the set of all subsets of X. $\mathcal{P}(X)$ is a ring with operations

$$A + B = (A \setminus B) \cup (B \setminus A) \qquad \text{and} \qquad A \times B = A \cap B.$$

Let R be the ring of all functions from X to $\mathbb{Z}_2$. For $A \in \mathcal{P}(X)$, define the function $F_A : X \to \mathbb{Z}_2$ by

$$F_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

Prove that the map $\mathcal{P}(X) \to R$ given by $A \mapsto F_A$ is a ring isomorphism.

Modules

Problem 7: Let R and S be rings. Let M be a left R-module, N a right S-module, and X an (R, S)-module.

(1) Show that $\mathrm{Hom}_S(N, X)$ is a left R-module under $(r\phi)(n) = r\phi(n)$.
(2) Show that $\mathrm{Hom}_R(M, X)$ is a right S-module under $(\phi s)(m) = \phi(m)s$.
(3) Show that there is an isomorphism

$$\mathrm{Hom}_R(M, \mathrm{Hom}_S(N, X)) \cong \mathrm{Hom}_S(N, \mathrm{Hom}_R(M, X))$$

of abelian groups.

Problem 8: Let R and S be rings. Let M be a flat right R-module. Let N be an (R, S)-bimodule. If N is flat as an S-module, show that $M \otimes_R N$ is flat as a right S-module.

Problem 9: Prove $\mathbb{Z}[i] \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{C}$ as rings.

Fields

Problem 10: Let $f(x) = x^6 + 3 \in \mathbb{Q}[x]$. Show that $f(x)$ is irreducible over $\mathbb{Q}$, determine its splitting field K and the determine the structure of $\mathrm{Gal}(K/\mathbb{Q})$.

Problem 11: Let K be an algebraic extension of a field F. Let R be a ring such that $F \subseteq R \subseteq K$. Prove that R is a field.

Problem 12: Let $\mathbb{F}_p$ denote the finite field with p, a prime, elements. Let $f(x) = x^p - x + a \in \mathbb{F}_p[x]$ for a non-zero $a \in \mathbb{F}_p$. Show that $f(x)$ is irreducible over $\mathbb{F}_p$ and then show that the Galois group of its splitting field is cyclic. (Hint: show that if α is a root of $f(x)$ then so is $\alpha + 1$.)