ALGEBRA PRELIM—MAY 2024

Work two problems from each of the four sections, i.e., eight in total. Clearly indicate which two problems from each section are to be graded. Otherwise, we will grade problems 1, 2, 4, 5, 7, 8, 10 & 11. In grading, the problems will be weighted equally.

All rings are assumed to have a 1-element and all ring homomorphisms are assumed to preserve the 1-elements.

Groups

Problem 1: Determine, up to isomorphism, all groups of order $805 = 5 \cdot 7 \cdot 23$.

Problem 2: A subgroup M of a group G is a *maximal* subgroup if $M \neq G$ and there are no subgroups H of G with $M \subsetneqq H \subsetneqq G$. For a group G, the *Frattini subgroup* of G, denoted $\Phi(G)$, is the intersection of all the maximal subgroups of G. If G has no maximal subgroups, we set $\Phi(G) = G$. Prove that $\Phi(G)$ is a characteristic subgroup of G.

Problem 3: For $n \geq 3$, let S_n be the symmetric group on n letters and let $A_n \leq S_n$ be the alternating group.

(1) Let $\sigma \in S_n$. Prove that $\sigma \in A_n$ if and only if σ is a product of 3-cycles.

(2) If $\sigma, \tau \in S_n$ are 3-cycles, prove that $\langle \sigma, \tau \rangle$ is isomorphic to $C_3, C_3 \times C_3, A_4$ or A_5. (C_n denotes the cyclic group of order n.)

Rings

Problem 4: Let R be a commutative ring. A proper ideal I of R is called a *primary* ideal if, whenever $xy \in I$, then $x \in I$ or $y^n \in I$ for some positive integer n. An element $x \in R$ is called *nilpotent* if $x^m = 0$ for some positive integer m.

(1) Show that an ideal I of R is primary if and only if $R/I \neq \{0\}$ and every zero-divisor in R/I is nilpotent.

(2) Show that, if an ideal I of R is primary, then

$$\operatorname{Rad}(I) := \{x \in R \mid x^n \in I \text{ for some positive integer } n\}$$

is a prime ideal.

Problem 5: Let X be a non-empty set. Let $\mathcal{P}(X)$ be the set of all subsets of X. Define an addition and a multiplication on $\mathcal{P}(X)$ by

$$A + B = (A \setminus B) \cup (B \setminus A) \qquad \text{and} \qquad A \times B = A \cap B.$$

Show that $\mathcal{P}(X)$ is a commutative ring with identity.

Problem 6: Let R be a ring. Let $u \in R$ have a right inverse, i.e., there exists $x \in R$ such that $ux = 1_R$. Prove that the following are equivalent:

(1) u has more than one right inverse;
(2) u is a left zero divisor, i.e. there exists a non-zero $y \in R$ such that $uy = 0_R$; and
(3) u is not a unit.

Modules

Problem 7: Let R be an integral domain. An element m of an R-module M is called a *torsion* element if there exists a nonzero $r \in R$ such that $rm = 0$.

(1) If $I \subseteq R$ is a principal ideal, then prove that the R-module $I \otimes_R I$ has no torsion element other than zero.
(2) Let $R = \mathbb{Z}[x]$ and $I = (2, x)$. Show that the R-module $I \otimes_R I$ has a non-zero torsion element.

Problem 8: The $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z}$ is divisible and hence injective.

(1) Show that for every left R-module M, the abelian group $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is a right R-module under $(\phi r)(m) = \phi(rm)$.
(2) Show that a left R-module M is flat if and only if the right R-module $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is injective.

Problem 9: Let R be a ring, M a left R-module, and $M' \subseteq M$ a submodule.

(1) For a subset $I \subseteq R$, show that the set

$$(M' :_M I) := \{m \in M \mid Im \subseteq M'\}$$

is a subgroup of the abelian group M.
(2) Show that if I is a right ideal of R, then $(M' :_M I)$ is an R-submodule of M.

Fields

Problem 10: Let $f(x) = x^4 - x^2 - 1 \in \mathbb{Q}[x]$. Show that $f(x)$ is irreducible over $\mathbb{Q}$, determine its splitting field K and the determine the structure of $\mathrm{Gal}(K/\mathbb{Q})$.

Problem 11: Let F be a finite field and let L be the subfield of F generated be elements of the form x^3, for $x \in F$, i.e.,

$$L = \langle x^3 \mid x \in F \rangle.$$

If $L \neq F$, determine $|F|$.

Problem 12: Let $K/\mathbb{Q}$ be a Galois extension with $\mathrm{Gal}(K/\mathbb{Q}) \cong S_5$, the symmetric group on 5 letters. Prove that K is the splitting field of a degree 5 polynomial in $\mathbb{Q}[x]$.