

Complex Variables
Preliminary Exam
May 2023

Directions: Do all of the following problems. **Show all your work and justify your answers.**

Notation: \mathbb{C} — the complex plane; $\mathbb{D} := \{z : |z| < 1\}$ — the unit disk;
 $x = \Re(z)$ and $y = \Im(z)$ denote the real part of z and the imaginary part of z , respectively;
 $H(G) = \{f : G \rightarrow \mathbb{C} : f \text{ is analytic}\}$.

Problems

1. State and prove Schwarz's Lemma.
2. Find a conformal map from the slit half-disc $\{x + iy \in \mathbb{D} : y > 0\} \setminus \{iy : 0 \leq y \leq \frac{1}{2}\}$ onto the strip $\{x + iy : 0 < y < 1\}$.
3. Evaluate $\int_0^\infty \frac{x^{2/3}}{(x+3)(x^2+4)} dx$ using the Residue Theorem.
4. Suppose $0 < |a_n| < |a_{n-1}| < \dots < |a_0|$. Prove that $na_0 + a_1z + a_2z^2 + \dots + a_nz^n$ has no zeros in \mathbb{D} .
5. Suppose $f \in H(\mathbb{C})$ and f' is M -Lipschitz; that is, for all $z, w \in \mathbb{C}$, we have the inequality $|f'(z) - f'(w)| \leq M|z - w|$. Prove that f is a polynomial of degree at most 2.
6. Suppose f is meromorphic on $\mathbb{C} \cup \{\infty\}$.
 - a. Prove that f has at most finitely many poles.
 - b. Use part a to prove that f must be a rational function.
 - b. Prove that if f is one-to-one, then f must be a Möbius transformation.
7. If f is analytic on \mathbb{C} and $\Re f(x + iy) = e^x \sin(y)$, find $\Im f(x + iy)$.
8. Let G denote the right-half plane $\{x + iy : x > 0\}$. A G -automorphism is a one-to-one analytic function which maps G onto itself. Describe all G -automorphisms. That is, what form must a G -automorphism have? Why?