Complex Variables
Preliminary Exam
August 2025

Directions: Do all of the following problems. **Show all your work and justify your answers.**

Notation: $\mathbb{C}$ denotes the complex plane; $\mathbb{R}$ denotes the real line; $\mathbb{D} := \{z : |z| < 1\}$ denotes the unit disk; $H(G) = \{f : G \to \mathbb{C} : f \text{ is analytic on the region } G\}$.

Problems

1. Find a conformal map from $\{re^{i\theta} : r > 2, 0 < \theta < \pi/2\}$ onto $\mathbb{C} \setminus \{re^{i\theta} : r < 2, 0 < \theta < \pi\}$.

2. Evaluate $\displaystyle\int_0^\infty \frac{x^{1/3}}{(x+1)^2}\, dx$.

3. If f_n is a sequence of one-to-one functions converging to f in the metric space $H(\mathbb{D})$, show that f must be one-to-one or constant.

4. Suppose f is entire and the real part of f has no zeros. Prove that f must be a constant function.

5. State and prove the Casorati-Weierstrass Theorem.

6. Suppose $a \in G_1 \subset G_2$ and f_j is a one-to-one analytic function mapping G_j onto the left half-plane $\{x + iy : x < 0\}$ with $f_j(a) = -1$, for $j = 1, 2$. Prove that $|f_2'(a)| \leq |f_1'(a)|$, and the inequality is strict if and only if G_1 is a proper subset of G_2.

7. Suppose f is analytic on the square $S = \{x+iy : -1 < x < 1, -1 < y < 1\}$ and continuous on the closure of S. If $f(x + iy) = 0$ when $x = 1$, prove that $f(z) = 0$ for all $z \in S$.

8. Suppose $M > 0$ and G is a region in $\mathbb{C}$. A function $f : G \to \mathbb{C}$ is M-Lipschitz if for all $a, b \in G$, we have $|f(a) - f(b)| \leq M|a - b|$. Let $\mathscr{F} = \{f : G \to \mathbb{C} : f \text{ is analytic and } M\text{-Lipschitz}\}$. Prove $\mathscr{F}$ is a normal family.