Complex Analysis Prelim

May 1997

Notation: Throughout this exam the open unit disk, $\{z : |z| < 1\}$, is denoted by $\mathbb{D}$. We write $\overline{\mathbb{D}}$ for its closure $\{z : |z| \leq 1\}$. $\mathbb{C}$ is the complex plane, and $\mathbb{R}$ is the reals. **Do all eight problems.**

1. Suppose f is a complex-valued function on $\mathbb{D}$. Write down four distinct conditions that are equivalent to "f is analytic on $\mathbb{D}$".

2. Find a conformal map f from the region $R = \{z = x + iy : |x| < 1, y > 0\}$ onto the upper half plane, $\mathbb{H} = \{z = x + iy : y > 0\}$, such that $f(-1) = -1, f(0) = 0, f(1) = 1$.

3. Let f be an analytic function on the unit disk $\mathbb{D}$ with only a finite number n of zeros in $\mathbb{D}$. Show that there is a polynomial p of degree n and an analytic function k on $\mathbb{D}$ such that $f(z) = e^{k(z)}p(z)$.

4. Suppose f is analytic in $\mathbb{D}$ and $|f(z)| \leq 1$ in $\mathbb{D}$. Show that for all $z \in \mathbb{D}$
$$|f'(z)| \leq \frac{1 - |f(z)|^2}{1 - |z|^2}.$$

5. Suppose f and g are meromorphic functions on $\mathbb{C}$ such that $g(z) = f(1/z)$ for $z \neq 0$. Show that f is a rational function.

6. Using the calculus of residues, prove:
$$\int_0^1 \frac{dx}{(x^2 - x^3)^{1/3}} = \frac{2\pi}{\sqrt{3}}.$$
(Hint: Consider a change of variable.)

7. How many solutions has $e^{z-2i} + 3z^8 + z^7 - 10z^6 + z^2 - z + 1 = 0$ in the open unit disk?

8. Suppose $\{f_n\}$ is a sequence of functions that are analytic on $\mathbb{D}$. Suppose there exists a function f on $\mathbb{D}$ such that for each $z \in \mathbb{D}$ we have $f_n(z) \to f(z)$. Suppose also there exists $\delta > 0$ such that $|f_n(z)| \geq \delta$ for all $z \in \mathbb{D}$ and all $n > 0$. Show that f_n converges uniformly on compact subsets of $\mathbb{D}$ and that f is analytic on $\mathbb{D}$.