Complex Analysis **Prelimary Examation** August 2002

Notation:

Let $\mathbf{C}$ denote the set of complex numbers. Let $\mathbf{D} = \{z \in \mathbf{C} : |z| < 1\}$ and let $\mathbf{U} = \{z \in \mathbf{C} : \text{Im } z > 0\}$.

For G a region in $\mathbf{C}$ let $A(G)$ denote the set of functions which are analytic on G and $H(G)$ denote the set of functions which are harmonic on G.

Do all ten problems. Provide appropriate justification for all solutions.

1. Show that the equation $(z-2)^2 = e^{-z}$ has 2 distinct roots in $\{z : |z-2| \leq 1\}$.
2. Let $u \in H(\mathbf{C})$ satisfy $u(x, y) \geq -22$ for all $z = x + iy \in \mathbf{C}$. Show that u is constant.
3. Let $f \in A(\mathbf{D} \setminus \{0\})$. Show that if f is bounded, then $f \in A(\mathbf{D})$.
4. Let $h \in A(\mathbf{C})$. Suppose that $h(0) = 3 + 4i$ and $|h(z)| \leq 5$ on $\mathbf{D}$. Find $h'(0)$.
5. Let $S = \{z : 0 < \text{Im } z < 2\}$. Find a conformal map f which maps S to $\mathbf{D}$ such that $f(i) = 0$ and $f(\infty) = 1$.
6. State and prove the Casorati-Weierstrass Theorem.
7. Let f be analytic on $\mathbf{U}$. Suppose for each $\zeta \in [0, 3]$ we have $\lim_{z\to\zeta, z\in \mathbf{U}} f(z) = 0$. Prove that $f \equiv 0$.
8. Give an example of an infinite normal family of analytic functions on $\mathbf{D}$ which contains unbounded functions.
9. Let $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots \in A(\mathbf{D})$ and suppose that f is nonvanishing on $\mathbf{D}$. For $0 < r < 1$, compute the geometric mean of f, i.e., compute $\exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log\left|f(re^{i\theta})\right| d\theta\right)$.
10. Let $\{g_n\}$ be a sequence of entire functions, whose only zeros are real zeros. Suppose that $\{g_n\}$ converges uniformly on compact subsets of $\mathbf{C}$ to g and $g \not\equiv 0$. Prove that g has only real zeros.