Complex Analysis **Preliminary Examination** May 2003

Answer all questions completely. Calculators may not be used. Notation: $\mathbb{D} = \{z : |z| < 1\}$, $H(G) = \{f : f \text{ is analytic on the region } G\}$.

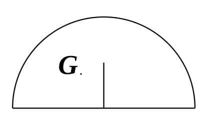
1. Suppose that $f : \mathbb{D} \to \mathbb{D}$ is holomorphic, with f having a zero of order at least n at $z = 0$, where $n \geq 1$. Show that $\left|f^{(n)}(0)\right| \leq n!$.

2. Suppose that $\mathcal{F} \subset H(\mathbb{D})$. Let $A = \{z : \frac{1}{2} < |z| < 1\}$ and define $\mathcal{F}_A = \{f|_A : f \in \mathcal{F}\}$. Show that if $\mathcal{F}_A$ is normal in $H(A)$ then $\mathcal{F}$ is normal in $H(\mathbb{D})$.

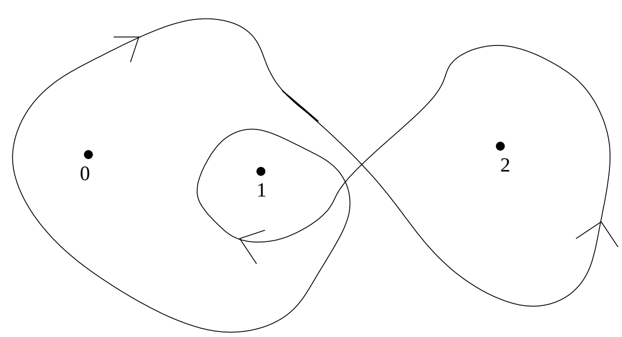
3. Suppose f is analytic in $\mathbb{D}$ and $|f(z)| \to 1$ as $|z| \to 1^-$. Show that the number of solutions (counting multiplicity) of $f(z) = \alpha$ is the same for all $\alpha \in \mathbb{D}$.

4. Describe the branches of $z \mapsto z^{\frac{1}{4}}$ on $G = \mathbb{C} \setminus (-\infty, 0]$. How many branches are there? Which is the principal branch? What is the range of each branch? Describe the associated Riemann surface.

5. Let $G = \{z \in \mathbb{D} | \text{Im } z > 0\} \setminus \{iy \mid 0 < y \leq \frac{1}{2}\}$. See the figure below. Give an explicit one-to-one conformal map which maps G onto $\mathbb{D}$.



6. Let $f(z) = \dfrac{1}{z^2(z-1)(z-2)}$. Find $\displaystyle\int_\gamma f(z)\,dz$ where γ is as pictured below:



7. Find the fallacy in the following argument:

Let m and n be two arbitrary integers. Then

$$e^{2m\pi i} = e^{2n\pi i};$$

hence,

$$\left(e^{2m\pi i}\right)^i = \left(e^{2n\pi i}\right)^i.$$

It follows that

$$e^{-2m\pi} = e^{-2n\pi}.$$

Since $-2m\pi$ and $-2n\pi$ are both real, $-2m\pi = -2n\pi$. Therefore,

$$m = n.$$

8. If u is a positive harmonic function on the ball $\{z \mid |z-a| < R\}$, show that

$$\frac{1}{3}u(a) \leq u(z) \leq 3u(a)$$

for $|z-a| \leq \frac{R}{2}$.