

Do all problems. Present adequate work to justify your answers.

Notation: $B(0; r) = \{z \in \mathbb{C} : |z| < r\}$, $\mathbb{D} = B(0; 1)$, $\text{ann}(a; \alpha, \beta) = \{z \in \mathbb{C} : \alpha < |z - a| < \beta\}$

1. Let G be the region in the first quadrant bounded by the line segment $[0, 1]$ and the arc of the circle which passes through 0 and 1 and which is tangent to the line $\text{Re} z = \text{Im} z$ at $z = 0$. Construct a one-to-one, conformal map of G onto \mathbb{D} .
2. Consider the rational function $f(z) = \frac{z^2 - 2z}{z(1-z)(2-z)^2}$.
 - a) Classify all of the singularities of f , including the singularity at ∞ .
 - b) Classify all of the singularities of $f(z^2)$, including the singularity at ∞ .
 - c) Find the Laurent expansion of f on the annulus $\text{ann}(0; 1, 2)$.
3. State and prove Liouville's Theorem.
4. Let $\mathbb{D}' = \mathbb{D} \setminus \{0\}$ and let $\mathcal{A}(\mathbb{D}')$ denote the set of analytic functions on \mathbb{D}' . Let $\{f_n\} \subset \mathcal{A}(\mathbb{D}')$ and $f \in \mathcal{A}(\mathbb{D}')$ such that f_n converges to f in the topology of local uniform convergence on compacta. Let $\sum_{k=-\infty}^{\infty} a_k^{(n)} z^k$ be the Laurent series expansion of f_n on \mathbb{D}' and $\sum_{k=-\infty}^{\infty} a_k z^k$ be the Laurent series expansion of f on \mathbb{D}' . Prove for each k that the sequence $\{a_k^{(n)}\}$ converges to a_k as $n \rightarrow \infty$.
5. Let f be analytic on $B(0; 10)$ such that for $z \in \partial B(0; 1)$ that $\text{Im} f(z) = \text{Im} z$. Find a representation for f if $f(0) = 1$.
6. Show for $\alpha > 1$ that $\alpha z^3 e^z = 1$ has exactly three roots in $B(0; 2)$.
7. For f analytic on \mathbb{C} we say that ζ is an *attractive fixed point* of f if ζ is a fixed point of f and if there exists a $\delta > 0$ such that $|f(z) - \zeta| < |z - \zeta|$ for $0 < |z - \zeta| < \delta$. Let $f(z) = z^2 - (2 - \frac{1}{2}i)z$. Find the attractive fixed points of f .
8. Let G be a region in \mathbb{C} and let f be analytic on G . Suppose there exists $\overline{B(a; r)} \subset G$ such that for $z \in \partial B(a; r)$, $|f(z)| = 1$. If $\inf_{z \in G} |f(z)| > 0$, show that f is constant.
9. Prove that the function $f(z) = \frac{1}{z^2}$ cannot be uniformly approximated by polynomials on the annulus $\text{ann}(0; 1, 2)$.
10. Let G be a region in \mathbb{C} and let $\mathcal{A}(G)$ denote the set of analytic functions on G . For any subset \mathcal{F} of $\mathcal{A}(G)$ let $\mathcal{F}' = \{f' : f \in \mathcal{F}\}$. If \mathcal{F} is a normal subset of $\mathcal{A}(G)$, prove that \mathcal{F}' is also normal.