

Do all problems. Present adequate work to justify your answers.

Notation: $B(0; r) = \{z \in \mathbb{C} : |z| < r\}$, $U = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$,
 $\mathbb{D} = B(0; 1)$, $\operatorname{ann}(a; \alpha, \beta) = \{z \in \mathbb{C} : \alpha < |z - a| < \beta\}$.

1. Evaluate the integral $\int_0^\infty \frac{\sin x}{x(1+x^2)} dx$.
2. Give an example of each of the following, or state why an example does not exist. (These are short answer questions.)
 - a) A function with a pole of order two at each integer.
 - b) An entire function which maps the open ball \mathbb{D} onto $\{z : \operatorname{Re} z \geq 0\}$.
 - c) An analytic, one-to-one function which maps \mathbb{D} onto \mathbb{C} .
 - d) A function with an essential singularity at $z = 0$.
 - e) A function which is analytic on \mathbb{C} , but which is not meromorphic on the Riemann sphere.
3. Find the Laurent expansion of $f(z) = \frac{1}{z^2(1-z)}$ in the given annulus.
 - a) $\operatorname{ann}(0; 0, 1)$.
 - b) $\operatorname{ann}(0; 1, \infty)$.
4. Let $S = \{z = re^{i\theta} : 0 < r < 3, |\theta| < \pi/6\} \setminus (0, 1]$. Find a one-to-one, conformal map f which takes S onto U such that $f(2) = i$.
5. Prove that $\Gamma(z+1) = z\Gamma(z)$ for $z \neq 0, -1, -2, \dots$.
6. Let $\sum_{n=0}^{\infty} a_n(z-1)^n$ be the series expansion for $\frac{1}{\cos z}$ about $z = 1$.
 - a) What is the radius of convergence of the series?
 - b) Does the series $\sum_{n=0}^{\infty} |a_n|$ converge?
7. Show that every meromorphic function on \mathbb{C} is the quotient of two entire functions.
8. How many roots does the equation $e^z = (e+1)z^5$ have in \mathbb{D} ?
9. Suppose that f is an analytic map from a region Ω into a region G and suppose that u is harmonic on G . Show that $u \circ f$ is harmonic on Ω .
10. Suppose that f is an entire function and that $|f(z) + 1| > 1$ for all $z \in \mathbb{C}$. Show that f is constant.