Complex Variables
Preliminary Exam
August 2006

Directions: Do all of the following ten problems. **Show all your work and justify your answers.** Each problem is worth 10 points. $\Re(z)$ and $\Im(z)$ denote the real part of z and the imaginar part of z, respectively.

1. Compute all values of $(1+i)^i$.

2. Prove that for all $z = x + iy$,
$$|\sinh z|^2 = \cosh^2 x - \cos^2 y.$$

3. Show that the function $u(x, y)$ is harmonic on $\mathbb{C} \setminus \{0\}$ and find its harmonic congugate $v(x, y)$ such that $v(1, 1) = 0$ if
$$u(x, y) = \frac{y}{x^2 + y^2}.$$

4. Find the radii of convergence of the Taylor expansions centered at $z_0 = 0$ of the following functions:
$$(\mathbf{a})\quad f(z) = \sum_{k=1}^{\infty} (5 + (-1)^k 3)^{2k} z^k \qquad (\mathbf{b})\quad g(z) = \sqrt{z - 1 + i} \qquad (\mathbf{c})\quad h(z) = \frac{\cos(iz)}{e^{2z}}$$

5. For the function
$$f(z) = \frac{1}{z^2(1 - z^2)},$$
find the Laurent expansion centered at $z_0 = 1$ that converges at $z = 4$. Determine the largest open set on which the series converges.

6. Calculate the residues at each isolated singularity in the extended complex plane $\overline{\mathbb{C}}$ of the functions
$$(\mathbf{a})\quad \frac{z^2}{(1 - z)^3} \qquad (\mathbf{b})\quad \frac{1}{\sin z}.$$

7. Use the Residue Calculus to evaluate the integral
$$\int_0^{+\infty} \frac{x^{\frac{1}{2}}}{(x+1)^2}\, dx.$$
Prove all your statements (in particular, if you claim a certain term tends to zero, you must show it does so).

8. **(a)** State Rouché's Theorem.
(b) Find the number of zeros of $f(z) = \operatorname{Log}(4 + z) - 7z^3 + 2z - 1$ in the unit disc $\mathbb{D} = \{z : |z| < 1\}$. Here Log denotes the principal branch of the logarithm.

9. Find a conformal mapping $w = f(z)$ from the upper half-plane $\{z : \Im(z) > 0\}$ onto the domain $D = \mathbb{C} \setminus (\{z : \Im(z) = 0,\ -\infty < \Re(z) \leq 0\} \cup \{z : \Re(z) = 0,\ |\Im(z)| \leq 2\})$.

10. Construct an entire function that has simple zeros at the points $z_n = 2n$, $n = 0$ and $n = 4, 5, \ldots$, and has no other zeros. Prove the convergence if this is necessary for your construction.