Complex Analysis — Preliminary Exam — May 2007

Instructions:

Do each of the following problems. Show all relevant steps which lead to your solutions.

Notation:

$\mathbb{C}$ denotes the complex plane.

For $z \in \mathbb{C}$, $\Re(z)$ and $\Im(z)$ denote the real and imaginay parts of z, respectively.

$\mathbb{D}$ denotes the open unit disk in $\mathbb{C}$, i.e., $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

$\mathbb{U}$ denotes the upper half-plane in $\mathbb{C}$, i.e., $\mathbb{U} = \{z \in \mathbb{C} : \Im(z) > 0\}$.

For a region $G \subset \mathbb{C}$, let $\mathcal{A}(G) = \{f : f \text{ is analytic on } G\}$.

1. (a) Find all solutions of the equation $e^{e^z} = 1$.

   (b) Factor the polynomial $p(z) = -32z - 32z^2 - 12z^3 - 2z^4$ given that $z = -2$ is a root.

2. Find and classify all isolated singular points of each of the following functions, including any isolated singular points which occur at the point of infinity:

   a. $e^{-z} \cos \dfrac{1}{z}$ 　　 b. $\dfrac{\cot z}{z^2}$

3. Prove that the function $f(z) = \displaystyle\sum_{n=1}^{\infty} n^{-z}$ converges for $\Re(z) > 1$ and represent its derivative in series form.

4. Use residue calculus to evaluate the integral $\displaystyle\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 - 2x + 10}\, dx$. Verify each step.

5. Let $A = \{z : \frac{1}{2} < |z| < 1\}$ and $f \in \mathcal{A}(A)$. Suppose there exists a sequence of polynomials $\{p_n\}$ such that $\{p_n\}$ converges to f in the topology of uniform convergence on compact subsets of A. Show that f can be extended to a function which is analytic on $\mathbb{D}$.

6. Let f be analytic on a region containing $\overline{\mathbb{D}}$ such that $|f(z)| \leq 1$ for $z \in \overline{\mathbb{D}}$. Show for all $a, b \in \mathbb{C}$ that $|af(0) + bf'(0)| \leq |a| + |b|$.

7. Determine how many roots $p(z) = z^4 + z + 1$ has in the first quadrant.

8. Find a conformal mapping from the sector $S = \{z : |z| < 4, 0 < \arg z < \pi/3\}$ slit along the radial segment $[0, e^{i\pi/6}]$ onto $\mathbb{U}$ such that $f(e^{i\pi/6}) = 0$, $f(2e^{i\pi/6}) = i$.

9. Let $f(z) = \dfrac{1}{1 + z^2 + z^4 + z^6 + z^8 + z^{10}}$. Find the radius of convergence of the Taylor series of f about $z = 1$.