Complex Variables
Preliminary Exam
May 2010

Directions: Do all of the following ten problems. **Show all your work and justify your answers.** Each problem is worth 10 points.

Notation: $\mathbb{C}$ — the complex plane; $\mathbb{D} := \{z : |z| < 1\}$ — the unit disk; $\Re(z)$ and $\Im(z)$ denote the real part of z and the imaginary part of z, respectively; $\operatorname{Log} z$ denotes the principal branch of the logarithm.

1. Compute all values of the following multi-valued expression: $(e^i)^i$.

2. Let $u(x, y)$ be harmonic on a domain $D \subset \mathbb{C}$ and let $v(x, y)$ be a harmonic conjugate of $u(x, y)$ on D.
(a) Prove that $u(x, y)v(x, y)$ is harmonic on D.
(b) Prove that if $x\, u(x, y)$ is harmonic on D then $u(x, y) = ay + b$, where a and b are constants.

3. Let G be a domain in $\mathbb{C}$, $a \in G$, and let $G_a = G \setminus \{a\}$. Suppose that f is a bounded analytic function on G_a. Prove that an isolated singularity of f at $z = a$ is removable.

4. Let f be an entire function. Suppose that there is a polynomial p such that for each $z \in \mathbb{C}$, $|f(z)| \le |p(z)|$. Show that f is also a polynomial.

5. **(a)** State any version of Runge's approximation theorem.
(b) Prove that there is a sequence of polynomials p_n such that $p_n(z) \to \sin z$ pointwise if $\Re z > 0$, $p_n(z) \to \cos z$ pointwise if $\Re(z) < 0$, and $p_n(z) \to 0$ pointwise if $\Re z = 0$.

6. Locate and classify for each of the functions all the singularities (including any singularity at $z = \infty$) as isolated or non-isolated. Further, classify the isolated singularities by type (removable, pole, essential):

(a) $\dfrac{1}{e^z - 1} - \dfrac{1}{z}$ **(b)** $\dfrac{1}{\operatorname{Log} z}$ **(c)** $z^2 \sin(1/z)$

7. Use the Residue Calculus to evaluate the integral
$$\int_0^\infty \frac{x^2\, dx}{x^4 + x^2 + 1}.$$

8. Let $A(\mathbb{D})$ be the set of analytic functions on the unit disk. Let $F = \{f \in A(\mathbb{D}) : f(0) = 1,\ f(\mathbb{D}) \subset \mathbb{C} \setminus (-\infty, 0]\}$. Use Schwarz's lemma to find
$$\max_{f \in F} |f'(0)|.$$

9. Find a conformal mapping $w = f(z)$ from the semi-disk $\mathbb{D}^+ := \{z \in \mathbb{D} : \Im(z) > 0\}$ onto itself with continuous extension to the boundary of $\mathbb{D}^+$ such that $f(-1) = 1$, $f(0) = i$, $f(1) = -1$.

10. Let f be a holomorphic function defined in a neighborhood of the closed disk $\overline{\mathbb{D}} = \{z : |z| \le 1\}$ such that $f(0) = 1$ and $|f(z)| > 1$ if $|z| = 1$. Prove that f has at least one zero in the unit disk $\mathbb{D}$.