Complex Analysis — Preliminary Exam — August 2011

Instructions:

Do each of the following problems. Show all relevant steps which lead to your solutions.

Notation:

$\mathbb{C}$ denotes the complex plane. $\mathbb{C}_\infty$ denotes the extended complex plane, i.e., $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$.

For $z \in \mathbb{C}$, $\Re z$ and $\Im z$ denote the real and imaginary parts of z, respectively.

$\mathbb{D}$ denotes the open unit disk in $\mathbb{C}$, i.e., $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

$B(a, r)$ denotes the open disk in $\mathbb{C}$ centered at a of radius r, i.e., $B(a, r) = \{z \in \mathbb{C} : |z - a| < r\}$.

$\operatorname{ann}(a; \alpha, \beta)$ denotes the open annulus in $\mathbb{C}$ centered at a of inner radius α and outer radius β, i.e., $\operatorname{ann}(a; \alpha, \beta) = \{z \in \mathbb{C} : \alpha < |z - a| < \beta\}$.

$\mathbb{U}$ denotes the upper half-plane in $\mathbb{C}$, i.e., $\mathbb{U} = \{z \in \mathbb{C} : \Im z > 0\}$.

For a region $G \subset \mathbb{C}$, let $\mathcal{A}(G) = \{f : f \text{ is analytic on } G\}$.

For a connected set $F \subset \mathbb{C}$, let $\mathcal{C}(F) = \{f \mid f : F \to \mathbb{C} \text{ and } f \text{ is continuous on } F\}$.

1. (a) Find all points $z \in \mathbb{C}$ where the function $f(z) = z \Re z$ is differentiable.

   (b) Find all points $z \in \mathbb{C}$ where the function $f(z) = z \Re z$ is analytic.

   (c) At each point $z \in \mathbb{C}$ where the function $f(z) = z \Re z$ is differentiable, find $f'(z)$.

2. Find an analytic function f such that $\Re\{f(z)\} = e^y \sin x$ for $z = x + iy$.

3. State and prove the Residue Theorem.

4. Let $F \in \mathcal{C}([0, 1])$. Define for $z \in \mathbb{C} \setminus [0, 1]$

$$f(z) = \int_0^1 \frac{F(t)}{t - z} dt.$$

   Prove, using the definiton of the derivative, that $f \in \mathcal{A}(\mathbb{C} \setminus [0, 1])$.

5. Find a function $f \in \mathcal{A}(\mathbb{D})$ such that f is non-constant and f has an infinite number of zeros in $\mathbb{D}$. Justify your steps.

6. Let $S = \{z : |\mathbb{R}\ z| > 1\}$. Let f be an entire function such that $f(\mathbb{C}) \subset S$. Prove that f is constant.

7. Find a one-to-one analytic map $w = f(z)$ from $\mathbb{D}$ onto $\left\{w : |\Im w| < \frac{\pi}{2}\right\} \setminus \{w : w \geq 0\}$.

8. Suppose $f \in \mathcal{A}(\mathbb{D})$ and satisfies $f(\mathbb{D}) \subset \mathbb{D}$ and $f(0) = 0$. Determine whether

$$F(z) \equiv \sum_{k=1}^{\infty} f(z^k) = f(z) + f(z^2) + f(z^3) + \cdots \in \mathcal{A}(\mathbb{D}).$$