Complex Analysis

Preliminary Exam

May 2011

Instructions:

Do each of the following problems. Show all relevant steps which lead to your solutions.

Notation:

$\mathbb{C}$ denotes the complex plane. $\mathbb{C}_\infty$ denotes the extended complex plane, i.e., $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$.

For $z \in \mathbb{C}$, $\Re z$ and $\Im z$ denote the real and imaginary parts of z, respectively.

$\mathbb{D}$ denotes the open unit disk in $\mathbb{C}$, i.e., $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

$B(a, r)$ denotes the open disk in $\mathbb{C}$ centered at a of radius r, i.e., $B(a, r) = \{z \in \mathbb{C} : |z - a| < r\}$.

$\operatorname{ann}(a; \alpha, \beta)$ denotes the open annulus in $\mathbb{C}$ centered at a of inner radius α and outer radius β, i.e., $\operatorname{ann}(a; \alpha, \beta) = \{z \in \mathbb{C} : \alpha < |z - a| < \beta\}$.

$\mathbb{U}$ denotes the upper half-plane in $\mathbb{C}$, i.e., $\mathbb{U} = \{z \in \mathbb{C} : \Im z > 0\}$.

For a region $G \subset \mathbb{C}$, let $\mathcal{A}(G) = \{f : f \text{ is analytic on } G\}$.

For a connected set $F \subset \mathbb{C}$, let $\mathcal{C}(F) = \{f \mid f : F \to \mathbb{C} \text{ and } f \text{ is continuous on } F\}$.

1. (a) Find a power series expansion for $f(z) = \dfrac{1}{2z - z^2}$ about $z = 1$.

   (b) Find a Laurent series expansion for $g(z) = \dfrac{1}{z} + \dfrac{1}{z+2} + \dfrac{1}{(z-1)^2}$ which is valid for $\operatorname{ann}(0; 1, 2)$.

2. Let f be an entire function such that $|f(z)| \leq A + B|z|^k$ for $z \in \mathbb{C}$ where A, B, k are positive constants. Prove that f is a polynomial.

3. Suppose that f is a meromorphic function on $\{z : |z| \leq 2\}$ with double zeros at both $1 + i$ and at $1 + 2i$, a double pole at $1/2 - i$ and a simple pole at 1. Compute the complex line integral

$$\int_\gamma z \frac{f'(z)}{f(z)} \, dz ,$$

   where γ is the circle $\{z : |z| = 3/2\}$.

4. State and prove Morera's Theorem.

5. Let G be a region and suppose $\{f_n\}$ is a sequence of analytic functions in $\mathcal{A}(G)$ that converges uniformly on compact subsets of G to a function f, which is continuous on G. Prove that $f \in \mathcal{A}(G)$.

6. Let f be analytic and non-zero on a simply connected domain G. Prove that $\log |f(z)|$ is harmonic on G.

7. Let $f \in \mathcal{A}(\mathbb{D})$ satisfy $|f(z)| \leq \dfrac{1}{1 - |z|}$. Show that $|f'(0)| \leq 4$.

8. Given α such that $0 < \alpha < \pi$, find a conformal mapping f from the domain $G = \{z : \Im z < 0\} \setminus \{e^{i\theta} : \pi \leq \theta \leq 2\pi - \alpha\}$ onto $\mathbb{U}$ such that $f(e^{-i\alpha/2}) = i$.

9. For $a, b > 0$, evaluate the integral $\displaystyle\int_0^\infty \frac{\cos ax}{x^2 + b^2} dx$. Justify your steps.