Complex Variables
Preliminary Exam
May 2013

Directions: Do all of the following eight problems. **Show all your work and justify your answers.** Each problem is worth 10 points.

Notation: $\mathbb{C}$ — the complex plane; $\mathbb{Z}$ — the set of integers; $\mathbb{D} := \{z : |z| < 1\}$ — the unit disk; $\Re(z)$ and $\Im(z)$ denote the real part of z and the imaginary part of z, respectively.

1. Let $f(z) = e^z$.
  (a) Use the Cauchy-Riemann Equations to prove that $f(z)$ is analytic on $\mathbb{C}$.
  (b) Prove that $f(z)$ is conformal at every point $z \in \mathbb{C}$.
  (c) Prove that $f(z)$ is one-to-one on the domain D, where

$$D := \{z = x + iy : -\infty < x < \infty,\ x < y < x + 2\pi\}.$$

2. **(a)** State Liouville's Theorem.
  (b) Show that there is no non-constant bounded analytic function on $\mathbb{C} \setminus \mathbb{Z}$.
  (c) Give an example of a function $f(z)$ which is analytic on $\mathbb{C} \setminus \mathbb{Z}$ but is not entire.

3. Let

$$f(z) = \cot z + \cos\left(\frac{1}{1-z}\right) - \frac{1}{z}.$$

Locate and classify all the singularities of $f(z)$ (including any singularity at $z = \infty$) as isolated or non-isolated. Further, classify the isolated singularities by type (removable, pole, essential). Calculate the residues of $f(z)$ at its poles.

4. Let

$$f(z) = \frac{cz^2 - cz + 1}{z^2(z-1)},$$

where $c \in \mathbb{C}$ is constant.
  (a) Find the principal part of the Laurent expansion of $f(z)$ convergent in the domain $D := \{z : 0 < |z| < 1\}$.
  (b) Find all values of c for which $f(z)$ has a primitive in D.

5. Let

$$f(z) = \begin{cases} \sin z & \text{if } \Im(z) \geq 0 \\ 1/\sin z & \text{if } \Im(z) < 0. \end{cases}$$

Prove that there is a sequence of polynomials $p_n(z)$, $n = 1, 2, 3, \ldots$ such that $p_n(z)$ converges to $f(z)$ point-wise on $\mathbb{C}$.

6. Use the Residue Theorem to evaluate the integral

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 - 2x + 10}\, dx.$$

7. Let $g(z)$ be analytic on the disk $\{z : |z| < 2\}$. Suppose that $g(z) \neq 0$ for all z such that $|z| = 1$ and $\Re\left(\frac{\sin(z^2)}{g(z)}\right) > 0$ for all z such that $|z| = 1$. Find the number of zeros (counting multiplicity) of $g(z)$ in the unit disk $\mathbb{D}$.

8. Let $\mathcal{A}(\mathbb{D})$ be the set of analytic functions on the unit disk. Let F be the set of all functions $f \in \mathcal{A}(\mathbb{D})$ such that $f(0) = 1$ and $|\arg(f(z))| < \pi/4$ for all $z \in \mathbb{D}$. Use Schwarz's lemma to find

$$\max_{f \in F} |f(1/2)|.$$