Complex Analysis **Preliminary Examination** August 2015

Answer all 8 questions.

Notation: $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$

1. State and prove Schwarz's Lemma. Discuss the case of equality.

2. Suppose f is analytic and has no zeros on a domain G. Prove that if $\dfrac{f(z)}{|f(z)|}$ is constant, then $f(z)$ is constant.

3. (a) Find a conformal map f from the domain $D_- = \mathbb{C} \setminus (-\infty, -1]$ onto the domain $D_+ = \mathbb{C} \setminus [1, +\infty)$ such that $f(0) = 0$ and $f'(0) < 0$.

   (b) Find a conformal map g from the domain $D_- = \mathbb{C} \setminus (-\infty, -1]$ onto the domain $D_+ = \mathbb{C} \setminus [1, +\infty)$ such that $g(0) = 0$ and $g'(0) > 0$.

4. Identify and classify all singularities of the following functions, including any singularities at infinity.

   (a) $\dfrac{1}{e^z + 1} + \dfrac{2\pi i}{z^2 + \pi^2}$

   (b) $z \cos\left(\dfrac{1}{z}\right)$

5. Suppose f is entire and $|f(z) \cdot f(-z)| \leq 1$ for all $z \in \mathbb{C}$. Prove that f has no zeros.

6. Suppose f is analytic on the unit disc $\mathbb{D}$ and $f\left(\dfrac{1}{n}\right) = -f\left(\dfrac{-1}{n}\right)$ for all integers $n \geq 2$. Show that f must be an odd function.

7. Let $f_n(z) = \dfrac{1}{1 + z + \frac{1}{2!}z^2 + \cdots + \frac{1}{n!}z^n}$ and let R_n be the radius of convergence of the power series expansion of f_n about 0. Prove that $R_n \to \infty$ as $n \to \infty$.

8. Use the Residue Theorem to evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{\cos(x)}{1 + x^2} dx$.