Complex Analysis **Preliminary Examination** May 2015

Answer all 8 questions.

Notation: $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, $B(a;r) = \{z \in \mathbb{C} : |z-a| < r\}$, $\text{Ann}(a;r_1;r_2) = \{z : r_1 < |z-a| < r_2\}$.

1. (a) State a version of Cauchy's Integral Formula.
   (b) State a version of Cauchy's Theorem.
   (c) Use Cauchy's Integral Formula to derive Cauchy's Theorem.

2. Suppose f is analytic on a domain G. Prove $\overline{z}\, f(z)$ is analytic if and only if $f(z) = 0$ for all $z \in G$.

3. Find a one-to-one conformal map f of the slit half-plane $\{z : \text{Re}\, z < 1\} \setminus (0,1]$ onto the upper half plane $\{z : \text{Im}\, z > 0\}$ such that on the boundary $f(0) = \infty$, $f(1+i) = -1$, and $f(1-i) = 1$.

4. Find the Laurent series centered at $z = 0$ of $\dfrac{4}{z^3 - 2z^2 - 3z}$ that converges uniformly on $|z| = 2$ and give the largest annulus on which it converges.

5. (a) Suppose f is entire and $a, b \in B(0;r)$. Evaluate $\displaystyle\int_\gamma \frac{f(z)}{(z-a)(z-b)}\, dz$, where $\gamma(t) = re^{it}$, $0 \le t \le 2\pi$.
   (b) Use part (a) to give a proof of Liouville's Theorem.

6. Let f be a nonconstant analytic function whose power series representation $\displaystyle\sum_{n=0}^{\infty} a_n z^n$ converges in $\mathbb{D}$. Suppose $f(z) = f(e^{i\alpha\pi} z)$ for all $z \in \mathbb{D}$ and some $\alpha \in \mathbb{R}$. Prove that α must be rational.

7. (a) Prove that there exists a sequence of polynomials $\{p_n\}$ which converges pointwise to $\dfrac{1}{z}$ on $\mathbb{C} \setminus (-\infty, 0]$.
   (b) Prove that no sequence of polynomials can converge to $\dfrac{1}{z}$ uniformly on compact subsets of $\text{Ann}(0;1;2)$.

8. Use the Residue Theorem to evaluate $\displaystyle\int_0^\infty \frac{x^{1/2}}{(1+x)^2}\, dx$.