Complex Variables
Preliminary Exam
May 2016

Directions: Do all of the following eight problems. **Show all your work and justify your answers.** Each problem is worth 10 points.

Notation: $\mathbb{C}$ — the complex plane; $\mathbb{D} := \{z : |z| < 1\}$ — the unit disk; $\Re(z)$ and $\Im(z)$ denote the real part of z and the imaginary part of z, respectively.

1. (a) Let $f : D \to \mathbb{C}$ be a complex-valued function defined on a domain $D \subset \mathbb{C}$. State the criterion of analyticity of f on D in terms of the Cauchy-Riemann Equations.
(b) Construct an example of a continuous function $f : \mathbb{C} \to \mathbb{C}$ that is differentiable on the given set E but is not analytic on any open subset of $\mathbb{C}$ or explain why such a construction is not possible if:
(α) E is a segment $[0,1] = \{z = x + i0 : 0 \le x \le 1\}$,
(β) E is the closed unit disk $\overline{\mathbb{D}} = \{z : |z| \le 1\}$.

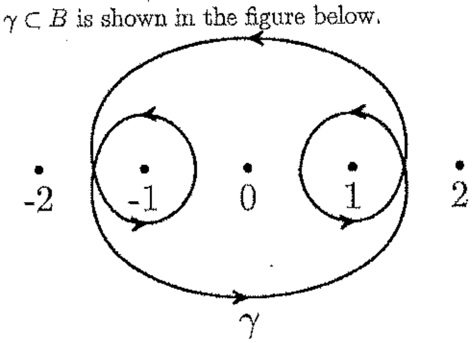
2. Let $\mathcal{F}$ be the class of functions $f(z)$ analytic at $z = 0$ and such that $|f^{(n)}(0)| \le n!$ for all $n \ge 0$. Let $R(f)$ denote the radius of convergence of the Taylor series of $f(z)$ centered at $z = 0$. Find

$$A = \inf_{f \in \mathcal{F}} R(f).$$

3. Let $f(z)$ be a function analytic in the disk $B = \{z : |z| < 100\}$ such that $f(0) = 3$ and $f(-1) + f(1) = 5$. Evaluate the integral

$$\int_\gamma \frac{f(z)}{e^{2\pi i z} - 1}\, dz,$$

where the rectifiable contour $\gamma \subset B$ is shown in the figure below.



4. Find all entire functions $f(z)$ such that $\Re(f(z)) + \Im(f(z)) \ne 2$ and $f(1)f(2)f(3)f(4) = 16$.

5. (a) State Rouché's Theorem.
(b) Find the number of roots of the equation $e^z = z$ in the horizontal strip $\{z : -10\pi < \Im z < 6\pi\}$. **Do not use Rouché's Theorem!**

6. Find a conformal mapping $\varphi(z)$ from the unit disk $\mathbb{D}$ onto the half-plane $H = \{w : \Im w > 0\}$ such that $\Re\varphi(0) = 0$ and $\varphi'(0) = -1$.

7. Let $f(z)$ be analytic in the disk $\{z : |z| < 2\}$. Suppose that a sequence of polynomials $p_n(z)$ converges to $f(z)$ uniformly on the unit circle $\mathbb{T} = \{z : |z| = 1\}$.
(a) Prove that $p_n(z)$ converges to $f(z)$ uniformly on the closed unit disk $\overline{\mathbb{D}} = \{z : |z| \le 1\}$.
(b) Show that $p_n(z)$ does not necessarily converge to $f(z)$ uniformly on a larger disk $\overline{\mathbb{D}}_r = \{z : |z| \le r\}$, $1 < r < 2$. (**Hint.** You may use an appropriate form of **Runge's Theorem**.)

8. Let $f : \mathbb{D} \to \mathbb{D}$ be analytic in the unit disk $\mathbb{D}$ with the Taylor expansion

$$f(z) = \alpha + a_n z^n + a_{n+1} z^{n+1} + \cdots$$

with some $0 < \alpha < 1$ and $n \ge 2$. Prove that

$$|a_n| \le 1 - \alpha^2.$$