Complex Variables
Preliminary Exam
August 2016

Directions: Do all of the following eight problems. **Show all your work and justify your answers.** Each problem is worth 10 points.

Notation: $\mathbb{C}$ — the complex plane; $\mathbb{D} := \{z : |z| < 1\}$ — the unit disk.

1. Let $f : \mathbb{D} \to \mathbb{D}$ be an analytic function with the Taylor expansion
$$f(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots \qquad (*)$$
For each integer $n \geq 2$, we define
$$f_n(z) = \frac{1}{n} \sum_{k=0}^{n-1} f(e^{i\frac{2\pi k}{n}} z)$$
Prove that $f_n(z)$ has the Taylor expansion of the form
$$f_n(z) = a_0 + a_n z^n + a_{2n} z^{2n} + a_{3n} z^{3n} + \cdots$$

2. Let $f : \mathbb{D} \to \mathbb{D}$ be a function analytic on $\mathbb{D}$ with the Taylor expansion as in $(*)$. Prove that
$$|a_n| \leq 1 \qquad \text{for all } n \geq 1.$$

3. **(a)** State Liouville's Theorem.
(b) Use Liouville's Theorem to prove the Fundamental Theorem of Algebra.

4. Use the Residue Calculus to evaluate the integrals
$$\text{(a)} \quad \int_0^\infty \frac{1 + x^2}{1 + x^4}\, dx \qquad\qquad \text{(b)} \quad \int_0^{2\pi} \frac{d\theta}{1 + \sin^2\theta}$$

5. **(a)** State a theorem containing the **Argument Principle.**
(b) Find the number of roots of the equation $z^5 + 14z + 2 = 0$ in the annulus $\{z : 3/2 < |z| < 2\}$. **Do not use the Argument Principle!**

6. Let Ω be a "crescent" domain bounded by circles $C_1 = \{z : |z| = 1\}$ and $C_2 = \{z : |z - \frac{1}{2}| = \frac{1}{2}$. Sketch the domain Ω. Find a conformal mapping $\varphi(z)$ from Ω onto the unit disk $\mathbb{D}$ such that $\varphi(-\frac{1}{3}) = 0$ and $\varphi(0) = 1$.

7. Locate and classify all singularities (including possible singularity at $z = \infty$) of:
$$\text{(a)} \quad \frac{z - 1}{z^4 - 3z^3 + 3z^2 - z} \qquad\qquad \text{(b)} \quad z \cot z \qquad\qquad \text{(c)} \quad z^2 \cos(1/z)$$

8. Let $f(z)$ be a non-constant meromorphic function on $\mathbb{C}$ with finite number of poles and infinitely many zeros. Prove that $z = \infty$ is an essential singularity of $f(z)$.