Complex Analysis **Preliminary Examination** May 2017

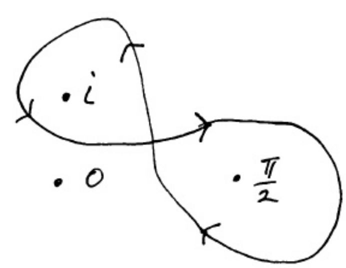
Answer all 8 questions. Show all work and completely explain your answers. Notation: $\mathbb{C}$ - complex plane, $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ - complex sphere, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ - unit disk, $B(a;r) = \{z \in \mathbb{C} : |z-a| < r\}$ - disk centered at a with radius $r > 0$, $\Re z$ - real part of z, $\Im z$ - imaginary part of z.

1. Suppose f is analytic on $B(a;r)$. Prove that f has a power series expansion about a with radius of convergence at least r.

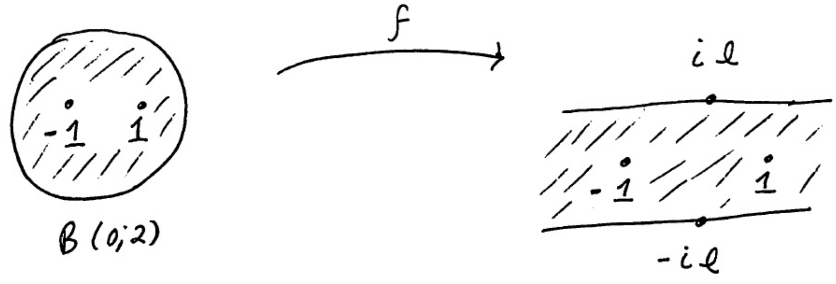
2. A real-valued function u is **harmonic** if $\Delta u = 0$. Suppose $u : \mathbb{D} \to \mathbb{D}$ is harmonic and $f : \mathbb{D} \to \mathbb{D}$ is analytic. Prove or disprove:
   (a) $f \circ u$ is analytic;
   (b) $u \circ f$ is harmonic.

3. Locate and classify all singularities (including any singularities at infinity) of
$$f(z) = \cos\left(\frac{\pi}{z}\right) + \tan\left(\frac{\pi}{2}z\right) - \frac{4}{\pi}\frac{1}{1-z^2}.$$

4. Evaluate the following integral $\displaystyle\int_\gamma \frac{\cos(z)}{z^2+1}\,dz$, where γ is the curve depicted below:



5. Find all values of ℓ so that there exists a conformal, one-to-one map $f : B(0;2) \to S(\ell)$ with $f(-1) = 1$ and $f(1) = -1$, where $S(\ell) = \{z \in \mathbb{C} : |\Im z| < \ell\}$. See the diagram below:



6. Find the number of zeros of the function $f(z) = 4z^{10} - e^z$ in $\mathbb{D}$. Are there any zeros of multiplicity greater than or equal to 2?

7. Suppose $\{f_n\}$ is a sequence of conformal, one-to-one maps from $\mathbb{D}$ onto the right half-plane $H_+ = \{z \in \mathbb{C} : \Re z > 0\}$. If $\{f_n\}$ converges to f uniformly on compact subsets of $\mathbb{D}$ and f is not one-to-one, find $\Re f(0)$.

8. Suppose f is entire and all coefficients of the power series for f centered at 0 are real. Prove that if f maps the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} : \Im z > 0\}$ onto a bounded subset of $\mathbb{C}$, then f must be constant.