Complex Analysis **Preliminary Examination** August 2017

Answer all 8 questions. Show all work and completely explain your answers. Notation: $\mathbb{C}$ - complex plane, $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ - complex sphere, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ - unit disk, $\Im z$ - imaginary part of z.

1. (a) Prove that $u(z) = \log|z|$ has a harmonic conjugate on the right half-plane $H_+ = \{z \in \mathbb{C} : \Re z > 0\}$ and find one.

   (b) Prove that $u(z) = \log|z|$ does not have a harmonic conjugate on $\mathbb{D} \setminus \{0\}$.

2. A **fixed point** of a function M is a solution of the equation $M(z) = z$. How many fixed points is it possible for a Möbius transformation to have? For each possibility, give an example of a Möbius transformation having that number of fixed points.

3. Evaluate $\displaystyle\int_0^\infty \frac{x^\alpha}{x^2+1}\,dx$, where $0 < \alpha < 1$.

4. Find a conformal map $f(z)$ from the domain $D = \overline{\mathbb{C}} \setminus \{z = t : -1 \le t \le 1\}$ onto the domain $\Omega = \overline{\mathbb{C}} \setminus \{z = it : -1 \le t \le 1\}$ with the property that the limit

$$\alpha = \lim_{z \to \infty} \frac{f(z)}{z}$$

   is positive. Find the value of α.

5. Suppose $f(z)$ is an analytic function on $\mathbb{D}$ with no zeros. Prove that there exists a sequence $\{z_n\}$ of points in $\mathbb{D}$ such that the sequence $\{|z_n|\}$ converges to 1 as $n \to \infty$ and the sequence $\{f(z_n)\}$ is bounded.

6. Let $f(z) = z + a_2 z^2 + \cdots$ be an analytic, one-to-one function on $\mathbb{D}$ such that $a_2 \neq 0$. Prove that there exist $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$ such that $e^{i\alpha} \in f(\mathbb{D})$ but $e^{i\beta} \notin f(\mathbb{D})$.

7. How many zeros does the polynomial $p(z) = z^4 + 3z^2 + z + 1$ have in the first quadrant? Explain completely.

8. Let $S = \{z \in \mathbb{C} : |\Im z| < 1\}$ and let $\mathcal{F} = \{f : \mathbb{D} \to S \text{ such that } f(0) = 0 \text{ and } f \text{ is analytic}\}$. Find

$$\max_{f \in \mathcal{F}} |f'(0)|.$$