**Complex Variables**
**Preliminary Exam**
May 2018

**Directions:** Do all of the following ten problems. **Show all your work and justify your answers.** Each problem is worth 10 points.

**Notation:** $\mathbb{C}$ — the complex plane; $\mathbb{D} := \{z : |z| < 1\}$ — the unit disk; $\Re(z)$ and $\Im(z)$ denote the real part of $z$ and the imaginary part of $z$, respectively.

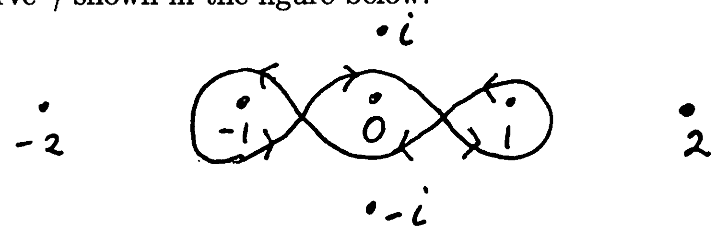
**1.** Let $f(z)$ be analytic in $\mathbb{D}$ with the Taylor expansion

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots$$

**(a)** Write two formulas, one in terms of derivatives and one in terms of integrals, for $a_n$.
**(b)** If, additionally, $f(z)$ satisfies the equation $f(ab) = f(a)f(b)$ for all $a, b \in \mathbb{D}$ and $1/16 < f(1/2) < 1/4$, find $f(z)$.

**2.** Let $f(z)$ be analytic on a domain $D$ and $f'(a) = 2$ at some point $a \in D$. Prove that
**(a)** there is $\varepsilon > 0$ such that $f(z)$ is one-to-one on the disk $\{z : |z - a| < \varepsilon\}$,
**(b)** $f(z)$ preserves angles between smooth arcs passing through $a$.

**3.** Suppose that $f(z)$ is meromorphic on $\mathbb{C}$ with only one pole in the disk $\{z : |z| < 3\}$ with the residue 3. Suppose further that $f(z)$ is analytic at the points $\pm 1$ and $\pm i$ and the radius of convergence at each of these points is 1. Find the integral $\int_\gamma f(z)\,dz$ over the closed curve $\gamma$ shown in the figure below.




**4.** Find the image of the square $Q = \{z : |\Re(z)| < 1, |\Im(z)| < 1\}$ under a mapping by analytic function $f : Q \to \mathbb{C}$ if $f(0) = 0$, $|f'(0)| = 1$, and $\arg f'(z) = \pi/4$ for all $z \in Q$.

**5.** Let

$$f(z) = \tan(\pi/(2z)) - \frac{2}{\pi}\frac{z}{1-z} + e^{\frac{1}{1+z}}.$$

Locate and classify all the singularities of $f$ (including any singularity at $z = \infty$) as isolated or non-isolated. Further, classify the isolated singularities by type (removable, pole, essential).

**6.** Prove that for any $|a| > \frac{1}{2}$ and integer $n \geq 3$, the equation $z + az^n = 1$ has at least one solution on the disk $\{z : |z| \leq 2\}$.

**7.** **(a)** State the Schwarz Reflection Principle.
**(b)** Find a conformal mapping $\varphi : Q_1 \to Q_1$ from the first quadrant $Q_1 = \{z : \Re(z) > 0, \Im(z) > 0\}$ onto itself such that $\varphi$ extends to be continuous on $\partial Q_1$ with $\varphi(i) = 0$, $\varphi(0) = 1$, $\varphi(\infty) = \infty$. Then describe possible extensions (via the Schwarz reflection principle) of $\varphi$ to the second quadrant $Q_2 = \{z : \Re(z) < 0, \Im(z) > 0\}$ and find possible values of the extended function at the point $z = -1 + i$; i.e. find possible values of $\varphi(-1 + i)$.

**8.** Let $\Omega$ be a bounded simply connected domain in the plane. Suppose that $g : \Omega \to \Omega$ is analytic and not the identity. Show that $g$ can have at most one fixed point.