**Complex Variables**
**Preliminary Exam**
August 2018

**Directions:** Do all of the following problems. **Show all your work and justify your answers.**

**Notation:** $\mathbb{C}$ — the complex plane; $\mathbb{D} := \{z : |z| < 1\}$ — the unit disk; $x = \Re(z)$ and $y = \Im(z)$ denote the real part of $z$ and the imaginary part of $z$, respectively.

**1.** Find a conformal mapping from the slit plane $\mathbb{C} \setminus (-\infty, 0]$ onto the slit plane $\mathbb{C} \setminus [0, \infty)$ such that $f(1) = -1$ and $f'(1) > 0$.

**2.** **(a)** If $f$ is analytic, prove that its real and imaginary parts satisfy the Cauchy-Riemann equations.
**(b)** Show that $u(x, y) = x^3 - 3xy^2 - 2x + 2$ is harmonic on $\mathbb{C}$ and find all harmonic conjugates of $u$.

**3.** Evaluate the integral

$$\int_0^\infty \frac{\cos x}{(x^2 + 1)(x^2 + 4)}\, dx.$$

**4.** Let

$$f(z) = \frac{1}{1 - z^2} \cos\left(\frac{\pi z}{z + 1}\right) + z(e^{\frac{1}{z}} - 1).$$

Locate and classify all the singularities of $f$ (including any singularity at $z = \infty$) as isolated or non-isolated. Further, classify the isolated singularities by type (removable, pole, essential). Calculate the residues of $f$ at its poles.

**5.** Find the number of complex numbers $z$ such that $|z| < 1$ and $e^z = z^4 + 5z^3 + 1$.

**6.** Let $f(z)$ be analytic in the upper half-plane $H_+ = \{z : \Im(z) > 0\}$ with $|f(z)| \leq \Im(z)$ for all $z \in H_+$.
**(a)** Prove that $f(z)$ extends to be analytic on $\mathbb{C}$.
**(b)** Find $f(i)$.

**7.** Prove that $f(z) = 2z + 6z^3 + 10z^5 + 14z^7 + 18z^9 + \ldots$ has a singularity on the unit circle.

**8.** **(a)** State the Casorati-Weierstrass Theorem.
**(b)** Suppose $f$ and $g$ are entire. Prove that if the composition $f \circ g$ is a polynomial, then both $f$ and $g$ are polynomials.