COMPLEX ANALYSIS
PRELIMINARY EXAMINATION
May 2019

Directions: Work all problems. Give as complete arguments as possible.
Notation: Im z denotes the imaginary part of z.

1. State and prove Morera's Theorem.

2. (a) How many zeros does the polynomial $P(z) = z^6 - 2z^5 + 3z^3 - 6z^2 + z - 2$ have on the boundary of the annulus $A = \{z \in \mathbb{C} \,|\, 1 < |z| < 2\}$?
   (b) How many zeros does it have inside the annulus $A = \{z \in \mathbb{C} \,|\, 1 < |z| < 2\}$?

3. Using a line integral and the Residues Theorem, compute
$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+9)^2}.$$

4. Suppose f is a holomorphic function on the unit disk with
   (a) $|f(z)| \leq 1$ for all z;
   (b) $f(0) = 0$.
   Is it true that
$$\text{Im } f'(0) \leq 1?$$
   If yes, prove your answer (do not just quote a theorem, but give a proof of the fact). If not, give a counterexample.

5. Let
$$f(z) = \frac{z - z\cos \pi z + \cot \pi z + e^{\frac{1}{z-1}}}{z^2}.$$
   Classify the singularities of f.

6. Show that the following series
$$\sum_{n=-\infty}^{\infty} e^{-\pi n^2 + 2\pi i n z}$$
   converges absolutely and uniformly on compact sets of the complex plane, and therefore defines an entire function (that is a holomorphic function on the complex plane).

7. Which of the following affine curves are Riemann surfaces?
$$X_1 = \{(z,w) \in \mathbb{C}^2 \,|\, w^3 = z^2 + 2z + 1\},$$
$$X_2 = \{(z,w) \in \mathbb{C}^2 \,|\, w^2 = z^3 - 3z^2 + 2z\},$$
$$X_3 = \{(z,w) \in \mathbb{C}^2 \,|\, z^3 + w^3 = 1\}.$$
   Motivate your answer.

8. (a) State (but do not prove) the Riemann Mapping Theorem and the Uniformization Theorem.
   (b) Define the Riemann zeta function and state (but do not prove!) the Riemann hypothesis.
   (c) State (but do not prove) the Weierstrass factorization theorem.