Directions: Work all problems. Give as complete arguments as possible.
Notation: Im \( z \) denotes the imaginary part of \( z \).

1. State and prove the Cauchy-Pompeiu Formula.

2. Show that \( u(x) = e^x \sin y \) is harmonic. Let \( z = x + iy \). Find a holomorphic function \( f(z) \), defined on the entire complex plane, such that \( u \) is the real part of \( f \).

3. Using a line integral and the Residue Theorem, compute the integral
\[
\int_0^\infty \frac{\cos x}{1 + x^2} dx.
\]

4. (a) State the Laurent series development theorem.
(b) Use the Laurent series representation to characterize isolated singularities of analytic functions.
(c) Does there exist a holomorphic function
\[
f : \mathbb{C} \setminus \{0\} \rightarrow \mathbb{C}
\]
such that for every \( r > 0 \), \( f(\{z \mid 0 < |z| < r\}) \) is the upper half-plane \( \text{Re} \, z > 0 \)?

5. Let \( K \subset \Omega \subset \mathbb{C} \), with \( K \) compact and \( \Omega \) open, such that every holomorphic function in a neighborhood of \( K \) can be approximated by holomorphic functions in \( \Omega \). Show that the open set \( \Omega \setminus K \) has no connected component whose closure is compact in \( \Omega \).

6. Find a conformal map
\[
f : \{z \in \mathbb{C} \mid 0 < \text{arg} \, z < \pi/2\} \rightarrow \{z \in \mathbb{C} \mid |z| < 1\}
\]
such that
\[
f \left( \frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2} \right) = 1/2 \text{ and } f' \left( \frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2} \right) < 0.
\]

7. Which of the following projective curves are Riemann surfaces?
\[
\bar{X}_1 = \{[Z_0,Z_1,Z_2] \in \mathbb{C}P^2 \mid Z_2^2 Z_0 + Z_1^2 Z_0 + Z_1 Z_0^2 + Z_0^3 = 0\}
\]
\[
\bar{X}_2 = \{[Z_0,Z_1,Z_2] \in \mathbb{C}P^2 \mid Z_0 Z_1^2 + Z_2^3 = 0\}.
\]

Motivate your answer.

8. Consider the Weierstrass p-function
\[
\wp(z) = \frac{1}{z^2} + \sum_{m,n \in \mathbb{Z} \atop (m,n) \neq (0,0)} \left( \frac{1}{(z - m - ni)^2} - \frac{1}{(m + in)^2} \right).
\]
Prove that there exist two complex numbers \( \omega_1 \) and \( \omega_2 \) such that the imaginary part of the quotient \( \omega_2/\omega_1 \) is positive, and \( \wp(z + \omega_1) = \wp(z) \) and \( \wp(z + \omega_2) = \wp(z) \), for all \( z \in \mathbb{C} \).