COMPLEX ANALYSIS
PRELIMINARY EXAMINATION
August 2019

Directions: Work all problems. Give as complete arguments as possible.
Notation: Im z denotes the imaginary part of z.

1. State and prove the Cauchy-Pompeiu Formula.

2. Show that $u(x) = e^x \sin y$ is harmonic. Let $z = x + iy$. Find a holomorphic function $f(z)$, defined on the entire complex plane, such that u is the real part of f.

3. Using a line integral and the Residue Theorem, compute the integral
$$\int_0^\infty \frac{\cos x}{1+x^2} dx.$$

4. (a) State the Laurent series development theorem.
(b) Use the Laurent series representation to characterize isolated singularities of analytic functions.
(c) Does there exist a holomorphic function
$$f : \mathbb{C}\backslash\{0\} \to \mathbb{C}$$
such that for every $r > 0$, $f(\{z \,|\, 0 < |z| < r\})$ is the upper half-plane Re $z > 0$?

5. Let $K \subset \Omega \subset \mathbb{C}$, with K compact and Ω open, such that every holomorphic function in a neighborhood of K can be approximated by holomorphic functions in Ω. Show that the open set $\Omega\backslash K$ has no connected component whose closure is compact in Ω.

6. Find a conformal map
$$f : \{z \in \mathbb{C} \,|\, 0 < \arg z < \pi/2\} \to \{z \in \mathbb{C} \,|\, |z| < 1\}$$
such that
$$f\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = 1/2 \text{ and } f'\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) < 0.$$

7. Which of the following projective curves are Riemann surfaces?
$$\bar{X}_1 = \{[Z_0, Z_1, Z_2] \in \mathbb{C}P^2 \,|\, Z_2^2 Z_0 + Z_1^3 + Z_1 Z_0^2 + Z_0^3 = 0\}$$
$$\bar{X}_2 = \{[Z_0, Z_1, Z_2] \in \mathbb{C}P^2 \,|\, Z_0 Z_1^2 + Z_2^3 = 0\}.$$
Motivate your answer.

8. Consider the Weierstrass p-function
$$\wp(z) = \frac{1}{z^2} + \sum_{\substack{m,n \in \mathbb{Z} \\ (m,n) \neq (0,0)}} \left(\frac{1}{(z-m-ni)^2} - \frac{1}{(m+in)^2}\right).$$
Prove that there exist two complex numbers ω_1 and ω_2 such that the imaginary part of the quotient ω_2/ω_1 is positive, and $\wp(z+\omega_1) = \wp(z)$ and $\wp(z+\omega_2) = \wp(z)$, for all $z \in \mathbb{C}$.