Directions: Do all of the following eight problems. **Show all your work and justify your answers.** Each problem is worth 10 points.

Notation: \( \mathbb{C} \) — the complex plane; \( \mathbb{D} := \{z : |z| < 1\} \) — the unit disk; \( \Re(z) \) and \( \Im(z) \) denote the real part of \( z \) and the imaginary part of \( z \), respectively.

1. (a) Let \( f : \Omega \to \mathbb{C} \) be differentiable (has complex derivative) on a domain \( \Omega \subset \mathbb{C} \). Prove that \( f \) satisfies the Cauchy-Riemann Equations on \( \Omega \).

(b) Construct an example of a continuous function \( f : \mathbb{D} \to \mathbb{C} \) that is differentiable (has complex derivative) on the given set \( E \subset \mathbb{D} \) but is not analytic on \( \mathbb{D} \) or explain why such construction is not possible if:

   (\( \alpha \)) \( E = (-1,1) = \{z = x + i0 : -1 < x < 1\} \),

   (\( \beta \)) \( E = \mathbb{D} \setminus (-1,1) \).

2. Let \( f(z) \) be a meromorphic function on \( \mathbb{C} \) with Taylor expansion

\[
f(z) = 1 - z^2 + z^4 - z^6 + z^8 - z^{10} + \cdots
\]

valid in a neighborhood of \( z = 0 \).

Prove that \( f(z) \) is analytic at \( z = 3 \). Then find the radius of convergence of the Taylor series \( f(z) = \sum_{n=0}^{\infty} c_n (z - 3)^n \) of \( f(z) \) centered at \( z = 3 \).

3. Use integration over an appropriate angular contour and Residue Theory to prove the following formula:

\[
\int_{0}^{\infty} \frac{x^{2m}}{1 + x^{2n}} \, dx = \frac{\pi}{2n} \frac{1}{\sin \left( \frac{2m+1}{2n} \pi \right)},
\]

where \( m \) and \( n \) are positive integers such that \( m < n \).

4. State and then prove the Fundamental Theorem of Algebra.

5. Show that the polynomial \( p(z) = z^4 + 2z^2 - z + 1 \) has at least one root in each quadrant.

6. Let \( \Omega \subset \mathbb{C} \) be a convex domain, \( a \in \Omega \), and let \( f(z) \) be analytic on \( \Omega \).

   (a) Prove that the function

\[
F(z) = (z - a) \int_{0}^{1} f(\gamma_2(t)) \, dt,
\]

where \( \gamma_2(t) = (1-t)a + tz, \quad 0 \leq t \leq 1 \), parameterizes the straight line segment from \( a \) to \( z \), is a primitive (antiderivative) of \( f(z) \) on \( \Omega \).

   (b) Suppose, in addition, that \( \Re(f'(z)) > 0 \) for all \( z \in \Omega \). Prove that \( f(z) \) is one-to-one (injective) on \( \Omega \).

7. (a) State Runge’s theorem (any version).

   (b) Prove or disprove the following statement:

   If a sequence of polynomials \( \{p_n(z)\}_{n=1}^{\infty} \) converges to a complex-valued function \( f(z) \) point-wise on the closed unit disk \( \overline{\mathbb{D}} = \{z : |z| \leq 1\} \), then \( f(z) \) is analytic on the open unit disk \( \mathbb{D} \).

8. (a) Define convergence of the infinite product \( \prod_{k=1}^{\infty} a_k \) of complex numbers \( a_k \).

   (b) Prove or disprove the following statement:

   If the infinite product \( \prod_{k=1}^{\infty} |a_k| \) of the moduli of complex numbers \( a_k \) converges, then the infinite product \( \prod_{k=1}^{\infty} a_k \) converges.