Complex Variables
Preliminary Exam
August 2020

Directions: Do all of the following eight problems. **Show all your work and justify your answers.** Each problem is worth 10 points.

Notation: $\mathbb{C}$ — the complex plane; $\mathbb{D} := \{z : |z| < 1\}$ — the unit disk; $\Re(z)$ and $\Im(z)$ denote the real part of z and the imaginary part of z, respectively.

1. **(a)** Let $f : \Omega \to \mathbb{C}$ be differentiable (has complex derivative) on a domain $\Omega \subset \mathbb{C}$. Prove that f satisfies the Cauchy-Riemann Equations on Ω.
(b) Construct an example of a continuous function $f : \mathbb{D} \to \mathbb{C}$ that is differentiable (has complex derivative) on the given set $E \subset \mathbb{D}$ but is not analytic on $\mathbb{D}$ or explain why such construction is not possible if:

(α) E is the interval $(-1, 1) = \{z = x + i0 : -1 < x < 1\}$,
(β) $E = \mathbb{D} \setminus (-1, 1)$.

2. Let $f(z)$ be a meromorphic function on $\mathbb{C}$ with Taylor expansion

$$f(z) = 1 - z^2 + z^4 - z^6 + z^8 - z^{10} + \cdots$$

valid in a neighborhood of $z = 0$.

Prove that $f(z)$ is analytic at $z = 3$. Then find the radius of convergence of the Taylor series $f(z) = \sum_{n=0}^{\infty} c_n (z-3)^n$ of $f(z)$ centered at $z = 3$.

3. Use integration over an appropriate angular contour and Residue Theory to prove the following formula:

$$\int_0^{\infty} \frac{x^{2m}}{1 + x^{2n}}\, dx = \frac{\pi}{2n} \frac{1}{\sin\left(\frac{2m+1}{2n}\pi\right)},$$

where m and n are positive integers such that $m < n$.

4. State and then prove the Fundamental Theorem of Algebra.

5. Show that the polynomial $p(z) = z^4 + 2z^2 - z + 1$ has at least one root in each quadrant.

6. Let $\Omega \subset \mathbb{C}$ be a convex domain, $a \in \Omega$, and let $f(z)$ be analytic on Ω.
(a) Prove that the function

$$F(z) = (z - a) \int_0^1 f(\gamma_z(t))\, dt,$$

where $\gamma_z(t) = (1-t)a + tz$, $0 \le t \le 1$, parameterizes the straight line segment from a to z, is a primitive (antiderivative) of $f(z)$ on Ω.
(b) Suppose, in addition, that $\Re(f'(z)) > 0$ for all $z \in \Omega$. Prove that $f(z)$ is one-to-one (injective) on Ω.

7. **(a)** State Runge's theorem (any version).
(b) Prove or disprove the following statement:
If a sequence of polynomials $\{p_n(z)\}_{n=1}^{\infty}$ converges to a complex-valued function $f(z)$ point-wise on the closed unit disk $\overline{\mathbb{D}} = \{z : |z| \le 1\}$, then $f(z)$ is analytic on the open unit disk $\mathbb{D}$.

8. **(a)** Define convergence of the infinite product $\prod_{k=1}^{\infty} a_k$ of complex numbers a_k.
(b) Prove or disprove the following statement:
If the infinite product $\prod_{k=1}^{\infty} |a_k|$ of the moduli of complex numbers a_k converges, then the infinite product $\prod_{k=1}^{\infty} a_k$ converges.