Complex Variables
Preliminary Exam
August 2021

Directions: Do all of the following eight problems. **Show all your work and justify your answers.** Each problem is worth 10 points.

Notation: $\mathbb{C}$ — the complex plane; $z = x + iy \in \mathbb{C}$; $\mathbb{D} := \{z : |z| < 1\}$ — the unit disk; $\Re(z)$ and $\Im(z)$ denote the real part of z and the imaginary part of z, respectively.

1. Describe and then graph the following subsets of the complex plane:
(a) $\{z : |z + i| + |z - i| = 4\}$,
(b) $\{z : \Re(z(1 - i)) < \sqrt{2}\}$,
(c) $\{z : \frac{\pi}{4} < \arg(z + i) < \frac{\pi}{2}\}$.

2. Prove that the function $u(x, y) = \log(x^2 + y^2)$ is harmonic in the domain $\Omega = \mathbb{D} \setminus \{0\}$. Then prove that $u(x, y)$ does not have a harmonic conjugate in Ω.

3. Let $f : \mathbb{D} \to \mathbb{D}$ be a function analytic on $\mathbb{D}$ with the Taylor expansion

$$f(z) = a_1 z + a_2 z^2 + a_3 z^3 + \cdots + a_n z^n + \cdots$$

Prove that

$$|a_n| \le 1 \qquad \text{for all } n \ge 1.$$

4. Use the Residue Calculus to evaluate the integrals

$$\text{(a)} \quad \int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9}\, dx \qquad\qquad \text{(b)} \quad \int_0^{2\pi} \frac{d\theta}{1 + \sin^2\theta}$$

5. How many roots (counted with multiplicity) does the function $f(z) = 6z^3 + e^z + 1$ have in the unit disk $\mathbb{D}$?

6. Let $\Omega = \{z : \Im z > 0\} \cap \{z : |z - 1| < 2\} \cap \{z : |z + 1| < 2\}$. Find a conformal mapping $\varphi(z)$ from Ω onto the upper half-plane $\mathbb{H} = \{z : \Im z > 0\}$.

7. Locate and classify all singularities (including possible singularity at $z = \infty$) of:

$$\text{(a)} \quad \frac{z + 1}{z^3 + 2z^2 + z} \qquad\qquad \text{(b)} \quad (z - 1)\tan(\pi z/2) \qquad\qquad \text{(c)} \quad z^3 \sin(1/z)$$

8. Let $f(z) = \begin{cases} 1 & \text{if } \Re z \ge 0, \\ -1 & \text{if } \Re z < 0. \end{cases}$

Determine whether the following statements are true or false. Justify your answers.

(a) There is a sequence of polynomials $p_n(z)$, $n = 1, 2, \ldots$, such that $p_n(z)$ converges to $f(z)$ point-wise on $\mathbb{C}$.

(b) There is a sequence of polynomials $p_n(z)$, $n = 1, 2, \ldots$, such that $p_n(z)$ converges to $f(z)$ point-wise on $\mathbb{C}$ and $p_n(z)$ converges to $f(z)$ uniformly on the square $Q_+ = \{z = x + iy : 0 \le x \le 200,\ |y| \le 100\}$.

(c) There is a sequence of polynomials $p_n(z)$, $n = 1, 2, \ldots$, such that $p_n(z)$ converges to $f(z)$ point-wise on $\mathbb{C}$ and $p_n(z)$ converges to $f(z)$ uniformly on the square $Q_- = \{z = x + iy : -200 \le x \le 0,\ |y| \le 100\}$.