Directions: Do all of the following eight problems. Show all your work and justify your answers. Each problem is worth 10 points.

Notation: \( \mathbb{C} \) — the complex plane; \( z = x + iy \in \mathbb{C}; \mathbb{D} := \{z : |z| < 1\} \) — the unit disk; \( \Re(z) \) and \( \Im(z) \) denote the real part of \( z \) and the imaginary part of \( z \), respectively.

1. Describe and then graph the following subsets of the complex plane:
   (a) \( \{z : |z + i| + |z - i| = 4\} \),
   (b) \( \{z : \Re(z(1 - i)) < \sqrt{2}\} \),
   (c) \( \{z : \frac{\pi}{4} < \arg(z + i) < \frac{\pi}{2}\} \).

2. Prove that the function \( u(x, y) = \log(x^2 + y^2) \) is harmonic in the domain \( \Omega = \mathbb{D} \setminus \{0\} \). Then prove that \( u(x, y) \) does not have a harmonic conjugate in \( \Omega \).

3. Let \( f : \mathbb{D} \to \mathbb{D} \) be a function analytic on \( \mathbb{D} \) with the Taylor expansion
   \[
   f(z) = a_1 z + a_2 z^2 + a_3 z^3 + \cdots + a_n z^n + \cdots
   \]
   Prove that
   \[
   |a_n| \leq 1 \quad \text{for all } n \geq 1.
   \]

4. Use the Residue Calculus to evaluate the integrals
   (a) \( \int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9} \, dx \)
   (b) \( \int_0^{2\pi} \frac{d\theta}{1 + \sin^2 \theta} \)

5. How many roots (counted with multiplicity) does the function \( f(z) = 6z^3 + e^z + 1 \) have in the unit disk \( \mathbb{D} \)?

6. Let \( \Omega = \{z : \Im(z) > 0\} \cap \{z : |z - 1| < 2\} \cap \{z : |z + 1| < 2\} \). Find a conformal mapping \( \varphi(z) \) from \( \Omega \) onto the upper half-plane \( \mathbb{H} = \{z : \Im(z) > 0\} \).

7. Locate and classify all singularities (including possible singularity at \( z = \infty \)) of:
   (a) \( \frac{z + 1}{z^3 + 2z^2 + z} \)
   (b) \( (z - 1) \tan(\pi z/2) \)
   (c) \( z^3 \sin(1/z) \)

8. Let \( f(z) = \begin{cases} 1 & \text{if } \Re(z) \geq 0, \\ -1 & \text{if } \Re(z) < 0. \end{cases} \)

   Determine whether the following statements are true or false. Justify your answers.
   (a) There is a sequence of polynomials \( p_n(z), n = 1, 2, \ldots, \) such that \( p_n(z) \) converges to \( f(z) \) point-wise on \( \mathbb{C} \).
   (b) There is a sequence of polynomials \( p_n(z), n = 1, 2, \ldots, \) such that \( p_n(z) \) converges to \( f(z) \) point-wise on \( \mathbb{C} \) and \( p_n(z) \) converges to \( f(z) \) uniformly on the square \( Q_+ = \{z = x + iy : 0 \leq x \leq 200, |y| \leq 100\} \).
   (c) There is a sequence of polynomials \( p_n(z), n = 1, 2, \ldots, \) such that \( p_n(z) \) converges to \( f(z) \) point-wise on \( \mathbb{C} \) and \( p_n(z) \) converges to \( f(z) \) uniformly on the square \( Q_- = \{z = x + iy : -200 \leq x \leq 0, |y| \leq 100\} \).